Canakci I, Schroll S.

Extensions in Jacobian algebras and cluster categories of marked surfaces.

***Advances in Mathematics* 2017, 313, 1-49.**

DOI link to article:

https://doi.org/10.1016/j.aim.2017.03.016

Date deposited:

25/01/2018

Advances in Mathematics 313 (2017) 1–49

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Extensions in Jacobian algebras and cluster categories of marked surfaces ☆,☆☆

Ilke Canakci [a], Sibylle Schroll [b,*]

[a] Department of Mathematical Sciences, Durham University, Stockton Road, Durham DH1 3LE, United Kingdom

[b] Department of Mathematics, University of Leicester, University Road, Leicester LE1 7RH, United Kingdom

A R T I C L E   I N F O

Article history:
Received 6 February 2015
Received in revised form 10 March 2017
Accepted 13 March 2017
Communicated by Henning Krause

MSC:
primary 13F60, 16P10, 18G15, 18E30

Keywords:
Extensions
Marked Riemann surfaces
Jacobian algebras
Cluster categories

A B S T R A C T

In the context of representation theory of finite dimensional algebras, string algebras have been extensively studied and most aspects of their representation theory are well-understood. One exception to this is the classification of extensions between indecomposable modules. In this paper we explicitly describe such extensions for a class of string algebras, namely gentle algebras associated to surface triangulations. These algebras arise as Jacobian algebras of unpunctured surfaces. We relate the extension spaces of indecomposable modules to crossings of generalised arcs in the surface and give explicit bases of the extension spaces for indecomposable modules in almost all cases. We show that the dimensions of these extension spaces are given in terms of crossing arcs in the surface.

Our approach is new and consists of interpreting snake graphs as indecomposable modules. In order to show that our basis is a spanning set, we need to work in the associated cluster category where we explicitly calculate the middle terms of extensions and give bases of their extension spaces. We note

☆ With an appendix by Claire Amiot. Institut Fourier-UMR 5582, 100 rue des maths, 38402 Saint Martin d'Héres, France, *e-mail address:* claire.amiot@univ-grenoble-alpes.fr.

☆☆ This work was supported through the Engineering and Physical Sciences Research Council, grant number EP/K026364/1, UK.

* Corresponding author.
E-mail addresses: IlkeCanakci@lgmail.com (I. Canakci), schroll@le.ac.uk (S. Schroll).

http://dx.doi.org/10.1016/j.aim.2017.03.016

that not all extensions in the cluster category give rise to extensions for the Jacobian algebra.

1. Introduction

Cluster algebras were introduced by Fomin and Zelevinsky in 2002 in [18] in order to give an algebraic framework for the study of the (dual) canonical bases in Lie theory. This work was further developed in [3,19,20]. Cluster algebras are commutative algebras given by generators, the *cluster variables*, and relations. The construction of the generators is a recursive process from an initial set of data. In general, even in small cases, this is a complex process. However, there is a class of cluster algebras coming from surfaces [16,17] (see also [14,15]) where this process is encoded in the combinatorial geometry of surface triangulations. Surface cluster algebras are an important part of the classification of (skew-symmetric) cluster algebras in terms of mutation type, namely almost all cluster algebras of finite mutation type are surface cluster algebras [13].

Cluster algebras from surfaces have been widely studied via the combinatorial geometry of the corresponding surfaces [13,16,17,32,33]. The same holds true for the associated cluster categories and Jacobian algebras. An important example of this is the crossing of two arcs in a surface. In the case of cluster algebras this gives rise to a multiplication formula for the corresponding cluster variables [34]. In the cluster category, the number of crossings of two arcs gives the dimension of the extension space between the associated indecomposable objects [42,38]. For Jacobian algebras of surfaces where all marked points lie in the boundary, in [6], building on [5] and [2], Auslander–Reiten sequences have been given in terms of arcs in the surface. In general, however, there has so far been no link between arbitrary crossings of arcs in the surface and the extensions between indecomposable modules in the Jacobian algebra.

The Jacobian algebras under consideration are gentle algebras and their indecomposable modules, given by strings and bands, correspond to curves and closed curves in the surface [2] (see [26,27,10,28] for a more general definition of Jacobian algebras via quiver with potential and [21] for classification of their representation type). Gentle algebras form a special class of algebras, for example, this class is closed under tilting and derived equivalence [40] and [41]. They are part of the larger family of string algebras which are an important family of algebras of tame representation type whose representation theory is well-understood. For example, their Auslander–Reiten structure has been determined [5] and in [11,25] the morphisms between indecomposable modules are completely described. However, a complete description of the extensions between indecomposable modules is not known.

In the present paper, we describe extension spaces of string modules over gentle Jacobian algebras. Furthermore, we show that in analogy with the cluster category, in most cases, the number of crossings of two arcs still gives rise to a dimension formula of the extension space between the corresponding indecomposable modules in the Jacobian algebra by explicitly constructing a basis. However, not every crossing contributes to this dimension. We characterise exactly which crossings contribute and which do not. We do this by introducing a new approach, consisting of using the snake graph calculus developed in [7] to explicitly construct extensions resulting from crossing arcs. This gives a lower bound on the dimensions of the extensions spaces in the Jacobian algebra. In order to obtain an upper bound, we work in the cluster category. There we show explicitly how, in many cases, the four arcs in the surface, resulting from the smoothing of a crossing of two not necessarily distinct arcs, give rise to two extensions in the cluster category.

For a surface cluster algebra, the cluster variables are in bijection with arcs in the surface [16]. Moreover, in a given triangulation, each arc corresponds to a combinatorial object called a *snake graph* [31,32,37]. Snake graphs have proven to be an important element in the understanding of surface cluster algebras, for example, in [33] snake graphs (and band graphs) were used to show that certain collections of loops and (generalised) arcs comprise vector space bases for surface cluster algebras. Snake graphs have also been instrumental in the proof of the positivity conjecture for surface cluster algebras [32]. Note that the conjecture has since been proved for all skew-symmetric cluster algebras [29].

If two (generalised) arcs γ_1 and γ_2 in a marked surface $(S, \mathcal{M})$ cross then the geometric operation of smoothing the crossing is given by locally replacing the crossing $\times$ with the pair of segments $\asymp$ or with the pair of segments $\supset\subset$. This gives rise to four new arcs γ_3, γ_4 and γ_5, γ_6 corresponding to the two different ways of smoothing the crossing. The corresponding elements $x_{\gamma_1}, \ldots, x_{\gamma_6}$ in the cluster algebra satisfy the so-called *skein relations* given by $x_{\gamma_1} x_{\gamma_2} = y_- x_{\gamma_3} x_{\gamma_4} + y_+ x_{\gamma_5} x_{\gamma_6}$ where y_-, y_+ are some coefficients [34].

Suppose from now on that $(S, \mathcal{M})$ is a marked surface such that all marked points are in the boundary of S and let T be a triangulation of $(S, \mathcal{M})$. All arcs are considered to be generalised, that is they might have self-crossings. The abstract snake graph calculus developed in [7] applies in this setting and gives a combinatorial interpretation in terms of snake graphs of the arcs resulting from the smoothing of two crossing arcs. We remark that we never actually smooth self-crossings as in [8], but instead in the case of a self-crossing we consider two copies of the same arc. We then use the combinatorial description in [7] to study the extension space over the associated Jacobian algebra $J(Q, W)$ and over the cluster category $\mathcal{C}(S, \mathcal{M})$ defined in [1] by giving explicit bases for these spaces in almost all cases. As mentioned above, the string modules over $J(Q, W)$ are in bijection with the arcs in the surface [2] not contained in T, and the arcs in the surface correspond in turn to snake graphs [32]. Therefore there is a correspondence associating a snake graph corresponding to an arc in $(S, \mathcal{M}, T)$ to the string module corresponding to the same arc and this defines a sign function on the snake graph, see Proposition 2.16.

Based on the snake graph calculus developed in [7], given two string modules, we define three types of crossings of modules corresponding to the three different types of crossings of the associated snake graphs. Namely, if, in the language of [8], the snake graphs cross with an overlap then we say that the corresponding string modules *cross in a module*. If the snake graphs cross with grafting and $s = d$ (see Section 2.4 for the definition of s and d) then we say that the corresponding string modules *cross in an arrow* and finally if the snake graphs cross with grafting and $s \neq d$ where s and d are parameters associated to the snake graphs then the corresponding string modules *cross in a 3-cycle*. Our first result is then to determine when two crossing string modules M_1 and M_2 give rise to a non-zero element in $\mathrm{Ext}^1_{J(Q,W)}(M_1, M_2)$.

Theorem 3.7. *Let M_1 and M_2 be two string modules (not necessarily distinct) over $J(Q, W)$ corresponding to arcs γ_1 and γ_2 in $(S, \mathcal{M})$. Then for a given crossing of M_1 and M_2 corresponding to a crossing of γ_1 and γ_2, there are string modules M_3 and M_4 obtained by 'smoothing the crossing' of γ_1 and γ_2 such that there exists an extension of M_1 by M_2*

(1) *with two non-zero middle terms given by M_3 and M_4 if and only if M_1 crosses M_2 in a module at this crossing,*
(2) *with one non-zero middle term given by M_3 if and only if M_1 crosses M_2 in an arrow at this crossing.*

When M_1 crosses M_2 in a 3-cycle, M_3 and M_4 do not give an extension of M_1 by M_2 corresponding to this crossing.

A geometric interpretation of the three different types of crossings in Theorem 3.7 is given in Remark 3.8.

Note that there is a direction in the crossing of modules, that is 'M_1 crosses M_2' is different from 'M_2 crosses M_1' and that this distinction does not appear in terms of crossings of the corresponding arcs, see Section 3.

We remark that Theorem 3.7 can be interpreted as skein relations for string modules and such skein relations have been announced by Geiss, Labardini and Schröer in the setting of Caldero–Chapoton algebras.

In the cluster category $\mathcal{C}(S, \mathcal{M})$, the indecomposable objects correspond to arcs and (non-contractible) closed loops in $(S, \mathcal{M})$ and therefore they are referred to as string and band objects, respectively, see [6]. It follows from the results on AR-triangles in [6] that all triangles between indecomposable objects in $\mathcal{C}(S, \mathcal{M})$ have at most two middle terms. In [42] it is shown that the dimension of the extension space of two string objects in the cluster category is equal to the number of crossings of the corresponding arcs. This suggests a close connection between the geometric crossing of arcs and the extension spaces.

Indeed, we show in Theorem 4.1 (see below) that the middle terms of triangles in the cluster category arise from smoothing the crossings of arcs in the surface in almost all cases. More precisely, consider the two pairs of arcs γ_3, γ_4 and γ_5, γ_6 obtained from smoothing a crossing of two arcs γ_1 and γ_2 (with a suitable orientation). We show that the pair γ_3, γ_4 always gives rise to an element in $\mathrm{Ext}_{\mathcal{C}}(\gamma_1, \gamma_2)$. We show that the other pair of arcs γ_5, γ_6 gives rise to an element in $\mathrm{Ext}_{\mathcal{C}}(\gamma_2, \gamma_1)$ if the crossing of γ_1 and γ_2 itself does not have a *self-crossing* (such a self-crossing is given by a *self-crossing overlap* in terms of the corresponding snake graphs. See Section 2, Definition 2.11 and Theorem 2.12 for the definition of a self-crossing overlap of snake graphs and the correspondence with crossings of arcs). We remark that an important factor in the proof of Theorem 4.1 is the geometric interpretation of Iyama–Yoshino reduction [22] given by Marsh–Palu [30].

Theorem 4.1. *Let γ_1 and γ_2 be two string objects (not necessarily distinct) in $\mathcal{C}(S, \mathcal{M})$ such that their corresponding arcs cross in $(S, \mathcal{M})$. Let $\gamma_3, \gamma_4, \gamma_5, \gamma_6$ be the string objects corresponding to the smoothing of a particular crossing of a suitable orientation of the corresponding arcs γ_1 and γ_2. Then there is a non-split triangle in $\mathcal{C}(S, \mathcal{M})$ given by*

$$\gamma_2 \longrightarrow \gamma_3 \oplus \gamma_4 \longrightarrow \gamma_1 \longrightarrow \gamma_2[1] \tag{1}$$

and if the crossing of γ_1 and γ_2 is not in a self-crossing overlap in some triangulation of $(S, \mathcal{M})$ then we obtain a non-split triangle given by

$$\gamma_1 \longrightarrow \gamma_5 \oplus \gamma_6 \longrightarrow \gamma_2 \longrightarrow \gamma_1[1]. \tag{2}$$

If any of $\gamma_3, \gamma_4, \gamma_5, \gamma_6$ are boundary arcs, the corresponding objects $\mathcal{C}(S, \mathcal{M})$ are zero objects.

We raise the question (Question 4.3) of what the middle terms are of those triangles in (2) above corresponding to the crossings of γ_1 and γ_2 that is a self-crossing overlap. That there must also be a non-zero extension $\mathrm{Ext}_{\mathcal{C}}(\gamma_2, \gamma_1)$ corresponding in some way to this crossing follows from the 2-Calabi Yau property of $\mathcal{C}(S, \mathcal{M})$ and from the result in [42] giving the dimension of the extension space in terms of the number of crossing of the arcs.

Theorem 3.8 together with Theorem 4.1 and the dimension formula in [42], give the following result for extensions in the Jacobian algebra. Here $\mathrm{Int}(\gamma, \delta)$ is the minimal number of intersections of two arcs γ and δ. We remark that we use the following convention: if $\gamma = \delta$ then $\mathrm{Int}(\gamma, \gamma) = 2m$ where m is the minimal number of self-crossings of γ.

Corollary 4.2. *Let M, N be two string modules over $J(Q, W)$ and let γ_M and γ_N be the corresponding arcs in $(S, \mathcal{M})$ such that γ_M and γ_N have no crossing with self-crossing overlap.*

(1) *A basis of $\mathrm{Ext}^1_{J(Q,W)}(M, N)$ is given by all short exact sequences arising from M crossing N in a module or an arrow and where the middle terms are as described in Theorem 3.7;*

(2) *We have*

$$\dim \operatorname{Ext}^1_{J(Q,W)}(M,N) + \dim \operatorname{Ext}^1_{J(Q,W)}(N,M) = \operatorname{Int}(\gamma_M, \gamma_N) - k - k'$$

where k (resp. k') is the number of times that M crosses N (resp. N crosses M) in a 3-cycle. In particular, if $M = N$ we have

$$2 \dim \operatorname{Ext}^1_{J(Q,W)}(M,M) = \operatorname{Int}(\gamma_M, \gamma_M) - 2k.$$

Acknowledgments: We would like to thank Ralf Schiffler for helpful conversations. We also would like to thank the referee for pointing out a gap in an earlier version of the proof of Theorem 4.1 and for the helpful comments and suggestions that they make.

2. Background on Jacobian algebras, cluster categories and snake graphs

Throughout let k be an algebraically closed field.

2.1. Bordered marked surfaces

In this section we follow [7,32] in our exposition. Let S be a connected oriented 2-dimensional Riemann surface with non-empty boundary. Let $\mathcal{M}$ be a finite set of marked points on S such that all marked points lie in the boundary of S and each boundary component contains at least one marked point. Call the pair $(S, \mathcal{M})$ a *(bordered) marked surface*. If S is a disc then let $|\mathcal{M}| \geq 4$.

Definition 2.1. A *generalised arc* in $(S, \mathcal{M})$ is a curve γ in S, considered up to homotopy, such that

(1) the endpoints of γ are in $\mathcal{M}$,
(2) except for the endpoints γ is disjoint from the boundary of S,
(3) γ does not cut out a monogon or a bigon.
The curve γ is called an *arc*, considered up to homotopy, if it satisfies (1), (2), (3), and
(4) γ does not cross itself, except that its endpoints might coincide.

A generalised arc may cross itself a finite number of times.

A *boundary segment* is the homotopy class of a curve that lies in the boundary and connects two (not necessarily distinct) neighbouring marked points on the same boundary component. Note that a boundary segment is not considered to be an arc. However, we sometimes do refer to it as a boundary arc.

Definition 2.2. For two arcs γ, γ' in $(S, \mathcal{M})$, let $\mathrm{Int}(\gamma, \gamma')$ be the minimal number of crossings of curves α, α' where α and α' range over the homotopy classes of γ and γ', respectively. We say that arcs γ, γ' are compatible if $\mathrm{Int}(\gamma, \gamma') = 0$.

Definition 2.3. A *triangulation* of $(S, \mathcal{M})$ is a maximal collection of pairwise compatible arcs. A *flip* of an arc γ in a triangulation T of $(S, \mathcal{M})$ replaces the arc γ with the unique arc γ' such that $T \setminus \{\gamma\} \cup \{\gamma'\}$ is a triangulation of $(S, \mathcal{M})$.

All triangulations of $(S, \mathcal{M})$ are connected by a series of flips.

Definition 2.4. Let γ_1 and γ_2 be generalised arcs such that γ_1 and γ_2 cross at a point x. We define the *smoothing of the crossing of* γ_1 *and* γ_2 *at the point* x to be the pairs of arcs $\{\alpha_1, \alpha_2\}$ and $\{\beta_1, \beta_2\}$ where

- $\{\alpha_1, \alpha_2\}$ is the same as $\{\gamma_1, \gamma_2\}$ except locally where the crossing $\times$ is replaced with the pair of segments $\asymp$,
- $\{\beta_1, \beta_2\}$ is the same as $\{\gamma_1, \gamma_2\}$ except locally where the crossing $\times$ is replaced with the pair of segments $\supset\subset$.

We remark that if we consider oriented arcs then the orientation of γ_1 and γ_2 permits to distinguish the set of arcs $\{\alpha_1, \alpha_2\}$ from the set of arcs $\{\beta_1, \beta_2\}$.

From now on we will not make a distinction between arcs and generalised arcs and we will simply call them arcs unless otherwise specified.

2.2. Gentle algebras from surface triangulations

In this section we recall the definition of gentle algebras and introduce some related notation which we will be using throughout the paper.

Let $Q = (Q_0, Q_1)$ be a quiver, denote by kQ its path algebra and for an admissible ideal I, let (Q, I) be the associated bound quiver. Denote by $\mathrm{mod}\, A$ the module category of finitely generated right A-modules of an algebra A.

Definition 2.5. An algebra A is *gentle* if it is Morita equivalent to an algebra kQ/I such that

(S1) each vertex of Q is the starting point of at most two arrows and is the end point of at most two arrows;
(S2) for each arrow α in Q_1 there is at most one arrow β in Q_1 such that $\alpha\beta$ is not in I and there is at most one arrow γ in Q_1 such that $\gamma\alpha$ is not in I;
(S3) I is generated by paths of length 2;
(S4) for each arrow α in Q_1 there is at most one arrow δ in Q_1 such that $\alpha\delta$ is in I and there is at most one arrow ε in Q_1 such that $\varepsilon\alpha$ is in I.

For $\alpha \in Q_1$, let $s(\alpha)$ be the start of α and $t(\alpha)$ be the end of α.

For each arrow α in Q_1 we define the formal inverse α^{-1} such that $s(\alpha^{-1}) = t(\alpha)$ and $t(\alpha^{-1}) = s(\alpha)$. A word $w = \varepsilon_1\varepsilon_2 \dots \varepsilon_n$ is a *string* if either ε_i or ε_i^{-1} is an arrow in Q_1, if $s(\varepsilon_{i+1}) = t(\varepsilon_i)$, if $\varepsilon_{i+1} \neq \varepsilon_i^{-1}$ for all $1 \leq i \leq n-1$ and if no subword of w or its inverse is in I. Let $s(w) = s(\varepsilon_1)$ and $t(w) = t(\varepsilon_n)$. Denote by $\mathcal{S}$ the set of strings modulo the equivalence relation $w \sim w^{-1}$, where w is a string.

A string w is a direct string if $w = \alpha_1\alpha_2 \dots \alpha_n$ and $\alpha_i \in Q_1$ for all $1 \leq i \leq n$ and w is an inverse string if w^{-1} is a direct string.

The terminology of string modules, in particular, the notions of hooks and cohooks were defined in [5]. However, the definitions of hooks and cohooks we give here differ slightly from the usual definitions. More precisely, our hooks and cohooks do not necessarily satisfy the maximality conditions on direct and inverse strings appearing in the standard literature.

Given a string w, define four substrings ${}_hw, w_h, {}_cw, w_c$ of w as follows:

We say ${}_hw$ is obtained from w by deleting a hook on $s(w)$ where

$$
{}_hw = \begin{cases} 0 & \text{if } w \text{ is an inverse string,} \\ {}_hw & \text{where } {}_hw \text{ is obtained from } w \text{ by deleting the first direct arrow in } w \\ & \text{and the inverse string preceding it.} \end{cases}
$$

We say ${}_cw$ is obtained from w by deleting a cohook on $s(w)$ where

$$
{}_cw = \begin{cases} 0 & \text{if } w \text{ is a direct string,} \\ {}_cw & \text{where } {}_cw \text{ is obtained from } w \text{ by deleting the first inverse arrow in } w \\ & \text{and the direct string preceding it.} \end{cases}
$$

We say w_h is obtained from w by deleting a hook on $t(w)$ where

$$
w_h = \begin{cases} 0 & \text{if } w \text{ is a direct string,} \\ w_h & \text{where } w_h \text{ is obtained from } w \text{ by deleting the last inverse arrow in } w \\ & \text{and the direct string succeeding it.} \end{cases}
$$

We say w_c is obtained from w by deleting a cohook on $t(w)$ where

$$
w_c = \begin{cases} 0 & \text{if } w \text{ is an inverse string,} \\ w_c & \text{where } w_c \text{ is obtained from } w \text{ by deleting the last direct arrow in } w \\ & \text{and the inverse string succeeding it.} \end{cases}
$$

Let T be a triangulation of $(S, \mathcal{M})$ with associated quiver with potential (Q, W) and let $J(Q, W)$ be the associated Jacobian algebra as defined in [26]. As recalled in the introduction, given a marked surface where all marked points lie in the boundary, this algebra coincides with the gentle algebra defined in [2]. Let $\mathcal{S}$ be the set of all strings in $J(Q, W)$. Given a string $w \in \mathcal{S}$ we denote by $M(w)$ the corresponding string module in

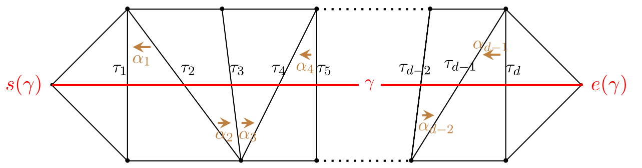


Fig. 1. Denoting the vertices in the quiver corresponding to an arc τ_i of the triangulation also by τ_i z, the string corresponding to the arc γ is given by $w(\gamma) = \tau_1 \overset{\alpha_1}{\leftarrow} \tau_2 \overset{\alpha_2}{\rightarrow} \tau_3 \overset{\alpha_3}{\rightarrow} \tau_4 \overset{\alpha_4}{\leftarrow} \tau_5 \cdots \tau_{d-2} \overset{\alpha_{d-2}}{\rightarrow} \tau_{d-1} \overset{\alpha_{d-1}}{\leftarrow} \tau_d$.

$J(Q, W)$. Note that $M(w) \simeq M(w^{-1})$, for a string w. Conversely, given a string module M we can associate to it a string (or its inverse). That is, there exists a string w_M such that $M \simeq M(w_M)$ in which case we also have $M \simeq M(w_M^{-1})$. The string corresponding to a simple module at vertex i of Q is denoted by i, that is, it is given by the single vertex i. Given an arc γ in the surface in [2] a string w_γ is associated to an orientation of γ. We denote $M(w_\gamma)$ the associated string module. The opposite orientation of γ gives rise to the inverse string $(w_\gamma)^{-1}$ and we have that $M(w_\gamma) \simeq M((w_\gamma)^{-1})$. Conversely, by [2] any string module M is associated to an arc γ_M in $(S, \mathcal{M}, T)$.

For the convenience of the reader we briefly recall in Fig. 1 the construction of a string given an orientated arc in a triangulated surface as defined in [2], see also [6].

2.3. *Cluster categories of marked surfaces*

Cluster categories were first introduced in [4] for acyclic quivers and independently in [9] for type A. Generalised cluster categories were defined in [1]. Namely, given a quiver with potential (Q, W) such that the Jacobian algebra $J(Q, W)$ is finite dimensional, denote by $\Gamma := \Gamma(Q, W)$ the associated Ginzburg dg-algebra. Consider the perfect derived category per Γ which is the smallest triangulated subcategory of the derived category $\mathcal{D}(\Gamma)$ containing Γ which is stable under taking direct summands and consider the bounded derived category $\mathcal{D}^b(\Gamma)$ of Γ. The generalised cluster category $\mathcal{C}(Q, W)$ is the quotient per $\Gamma/\mathcal{D}^b(\Gamma)$. It is shown in [1] that $\mathcal{C}(Q, W)$ is Hom-finite, 2-Calabi–Yau, the image of Γ in $\mathcal{C}(Q, W)$ is a cluster tilting object T_Γ, and the endomorphism algebra of T_Γ is isomorphic to the Jacobian algebra $J(Q, W)$. Furthermore, the categories $\mathcal{C}(Q, W)/T_\Gamma$ and mod $J(Q, W)$ are equivalent and the functor $\mathrm{Hom}_{\mathcal{C}(Q,W)}(T_\Gamma[-1], -)$ is the projection functor from $\mathcal{C}(Q, W)$ to mod $J(Q, W)$ [24].

Now let $(S, \mathcal{M})$ be a marked surface, T, T' triangulations of $(S, \mathcal{M})$, and let (Q, W) and (Q', W') be the quivers with potential associated with T and T', respectively. It follows from [16,23,26] that $\mathcal{C}(Q, W)$ and $\mathcal{C}(Q', W')$ are triangle equivalent and hence the cluster category is independent of the triangulation of $(S, \mathcal{M})$. We will thus denote the cluster category by $\mathcal{C}(S, \mathcal{M})$ or $\mathcal{C}$.

In [6] the cluster category $\mathcal{C}(S, \mathcal{M})$ associated to a surface with marked points on the boundary is explicitly described. In particular, a parametrization of the indecomposable objects of $\mathcal{C}(S, \mathcal{M})$ is given in terms of string objects and band objects. The string objects correspond bijectively to the homotopy classes of non-contractible curves in $(S, \mathcal{M})$ that are not homotopic to a boundary segment of $(S, \mathcal{M})$ and subject to the equivalence relations $\gamma \sim \gamma^{-1}$. The band objects correspond bijectively to the elements of $k^* \times \Pi_1^*(S, \mathcal{M})/\sim$ where $\Pi_1^*(S, \mathcal{M})$ are the invertible elements of the fundamental group of $(S, \mathcal{M})$ and where $\sim$ is the equivalence relation generated by $\gamma \sim \gamma^{-1}$ and cyclic permutation of γ.

Furthermore, it is shown in [6] that the AR-translation of an indecomposable object γ corresponds to simultaneously rotating the start and end points of γ in the orientation of $(S, \mathcal{M})$.

Unless otherwise stated we will not distinguish between arcs and the corresponding indecomposable objects in $\mathcal{C}(S, \mathcal{M})$.

The following theorem plays a crucial role in our results.

Theorem 2.6. *[42, Theorem 3.4] Let γ and δ be two (not necessarily distinct) arcs in $(S, \mathcal{M})$. Then*

$$\dim_k \operatorname{Ext}^1_{\mathcal{C}}(\gamma, \delta) = \operatorname{Int}(\gamma, \delta).$$

2.4. Snake graphs

In this section we state and prove two results relating to snake graphs, namely Propositions 2.13 and 2.16. These results form the basis of many of the proofs in sections 3 and 4. For the convenience of the reader we recall in this section all relevant results on snake graphs that we refer to in later sections. We define snake graphs associated to triangulations of surfaces as in [32] and [7,8]. Below, we closely follow the exposition in [8] adapting it to snake graphs associated to surface triangulations.

Let T be a triangulation of $(S, \mathcal{M})$ and γ be an arc in $(S, \mathcal{M})$ which is not in T. Choose an orientation of γ. We choose γ to be a representative in its homotopy class which transversally intersects the arcs of T such that no arc of T is crossed twice in succession. Let $\tau_{i_1}, \ldots, \tau_{i_d}$ be the arcs of T crossed by γ in the order given by the orientation of γ. Note that we choose γ to be a representative in its homotopy class such that γ has a minimal number of intersections with $\tau_{i_1}, \ldots, \tau_{i_d}$. It is however possible that $\tau_{i_j} = \tau_{i_k}$ for $j \neq k$. For an arc τ_{i_j}, let Δ_{j-1} and Δ_j be the two triangles in $(S, \mathcal{M}, T)$ that share the arc τ_{i_j} and such that γ first crosses Δ_{j-1} and then Δ_j. Note that each Δ_j always has three distinct sides, but that two or all three of the vertices of Δ_j might be identified. Let G_j be the graph with 4 vertices and 5 edges, having the shape of a square (with a fixed side length) with a diagonal that satisfies the following property: there is a bijection of the edges of G_j and the 5 distinct arcs in the triangles Δ_{j-1} and Δ_j and such that the diagonal in G_j corresponds to the arc τ_{i_j}. That is, G_j corresponds to the quadrilateral with diagonal τ_{i_j} formed by Δ_{j-1} and Δ_j in $(S, \mathcal{M}, T)$.

Given a planar embedding $\tilde{G}_j$ of G_j, we define the *relative orientation* $\operatorname{rel}(\tilde{G}_j, T)$ of $\tilde{G}_j$ with respect to T to be 1 or -1 depending on whether the triangles in $\tilde{G}_j$ agree or disagree with the (common) orientation of the triangles Δ_{j-1} and Δ_j in $(S, \mathcal{M}, T)$.

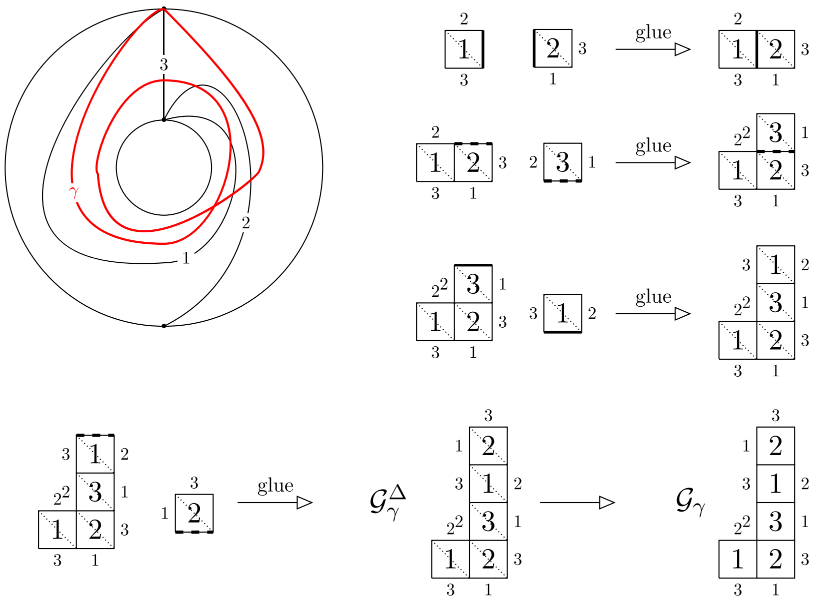


Fig. 2. Snake graph associated to the arc γ.

Using the notation above, the arcs τ_{i_j} and $\tau_{i_{j+1}}$ form two edges of the triangle Δ_j. Let σ_j be the third arc in this triangle. We now recursively glue together the tiles $G_1, \ldots, G_d$ one by one from 1 to d in the following way: choose planar embeddings of the G_j such that $\mathrm{rel}(\tilde{G}_j, T) \neq \mathrm{rel}(\tilde{G}_{j+1}, T)$. Then glue $\tilde{G}_{j+1}$ to $\tilde{G}_j$ along the edge labelled σ_j.

After gluing together the d tiles $G_1, \ldots, G_d$, we obtain a graph (embedded in the plane) which we denote by $\mathcal{G}^{\Delta}_{\gamma}$.

Definition 2.7. The *snake graph* $\mathcal{G}_\gamma$ associated to γ is obtained from $\mathcal{G}^{\Delta}_{\gamma}$ by removing the diagonal in each tile. If $\tau \in T$ then we define the associated snake graph $\mathcal{G}_\tau$ to be the graph consisting of one single edge with two distinct vertices (regardless of whether the endpoints of τ are distinct or not).

The labels on the edges of a snake graph, given by the corresponding arcs in the triangulation, are called *weights*. Sometimes snake graphs with weights are referred to as *labelled snake graphs*. See Fig. 2 for an example of a labelled snake graph associated to an arc.

The $d-1$ edges corresponding to the arcs $\sigma_1, \ldots, \sigma_{d-1}$ which are contained in two tiles are called *interior edges of* $\mathcal{G}_\gamma$. Denote this set by $\mathrm{Int}(\mathcal{G}_\gamma)$. The edges of $\mathcal{G}_\gamma$ not in $\mathrm{Int}(\mathcal{G}_\gamma)$ are called *boundary edges*. We define a subgraph $\mathcal{G}_\gamma[i, i+t]$, for $1 \leq i \leq d$ and for $0 \leq t \leq d-i$, to be the subgraph of $\mathcal{G}_\gamma$ consisting of the tiles $(G_i, \ldots, G_{i+t})$.

Let ${}_{SW}\mathcal{G}_\gamma$ (resp. $\mathcal{G}^{NE}_\gamma$) be the set containing the 2 elements corresponding to the south and west edge of G_1 (resp. the north and east edge of G_d). Define $\mathcal{G}_\gamma \setminus \mathrm{Pred}(\sigma_i) =$

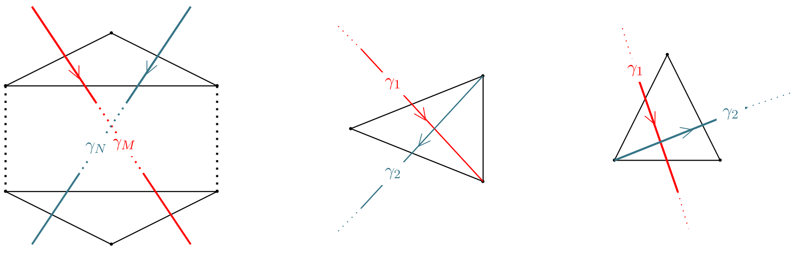


Fig. 3. The leftmost figure corresponds to an overlap crossing in terms of snake graphs and a module crossing in terms of string modules, the middle figure corresponds to grafting with $s = d$ in terms of snake graphs and an arrow crossing in terms of string modules, and the rightmost figure corresponds to grafting with $s \neq d$ in terms of snake graphs and a 3-cycle crossing in terms of string modules.

$\mathcal{G}_\gamma[i+1, d]$. If e is an edge in $\mathcal{G}_\gamma^{NE}$ then $\mathcal{G}_\gamma \backslash \mathrm{Pred}(e) = \{e\}$. Analogously, let $\mathcal{G}_\gamma \backslash \mathrm{Succ}(\sigma_i) = \mathcal{G}_\gamma[1, i]$. If e is an edge in ${}_{SW}\mathcal{G}_\gamma$ then $\mathcal{G}_\gamma \setminus \mathrm{Succ}(e) = \{e\}$.

If all tiles of a snake graph $\mathcal{G}_\gamma$ are in a row or a column, we call $\mathcal{G}_\gamma$ *straight* and we call it *zigzag* if no three consecutive tiles are straight.

Note that there is a notion of abstract snake graphs as combinatorial objects introduced in [7]. However, all snake graphs we consider here are snake graphs associated to arcs in triangulated surfaces as introduced above. In general, we will use the notation $\mathcal{G}$ if we do not need to refer to the associated arc or if the arc is clear from the context.

Definition 2.8. A *sign function* on a snake graph $\mathcal{G}$ is a function f from the set of interior edges $\{\sigma_1, \dots, \sigma_{d-1}\}$ to the set $\{+, -\}$ such that

(1) if three consecutive tiles G_{i-1}, G_i, G_{i+1} form a straight piece then $f(\sigma_{i-1}) = -f(\sigma_i)$,
(2) if three consecutive tiles G_{i-1}, G_i, G_{i+1} form a zigzag piece then $f(\sigma_{i-1}) = f(\sigma_i)$.

Extend the sign function to all edges of $\mathcal{G}$ by the following rule: opposite edges have opposite signs and the south side and east side of each tile have the same sign as do the north and west side of each tile.

Note that for every snake graph there are two sign functions, f and f' such that $f(\sigma_i) = -f'(\sigma_i)$, for each $1 \leq i \leq d-1$.

A crossing of two arcs γ_1, γ_2 has an interpretation in terms of the associated snake graphs as given in [7] and as further explored in [8]. Depending on the triangulation and the arcs, there are three different ways in which the two arcs can cross, see Fig. 3. In the first case, the arcs cross in what we refer to as an *overlap* since both arcs γ_1 and γ_2 cross at least one common arc in the triangulation and the crossing can be moved up to homotopy to any triangle adjoining an arc of the triangulation crossed by both γ_1 and γ_2. In the last two cases the crossing occurs in a single triangle and cannot be moved outside of this triangle by homotopy. We say that the arcs *cross in a triangle*.

In terms of snake graphs and using the terminology of [8] these crossings correspond to a crossing overlap, grafting with $s = d$ and grafting with $s \neq d$. Note that the integer

d corresponds to the number of tiles of the snake graph $\mathcal{G}_1$ corresponding to γ_1 and s is some integer $1 \le s \le d$ as defined in [8, Section 3.3] denoting the position at which the 'grafting' takes place. We will later see in Section 3 that these correspond to three types of module crossings, namely crossing in a module, arrow crossing, and 3-cycle crossing, respectively.

We start by defining an overlap of two snake graphs and a self-overlap of a snake graph.

Definition 2.9. [8, Section 2.5] Let $\mathcal{G}_1 = (G_1, G_2, \dots, G_d)$ and $\mathcal{G}_2 = (G'_1, G'_2, \dots, G'_{d'})$ be two snake graphs such that there exists two embeddings of graphs $i'_1 : \mathcal{G} \to \mathcal{G}_1$ and $i'_2 : \mathcal{G} \to \mathcal{G}_2$ such that, for $j = 1, 2$, the image $i_j(\mathcal{G})$ is either identical to $\mathcal{G}$, a 180° rotation of $\mathcal{G}$, or a reflection of $\mathcal{G}$ at one of the lines $y = x$ or $y = x$. In particular, the south west vertex of the first tile of $\mathcal{G}$ has to be mapped to the south west vertex of the first tile in $i_j(\mathcal{G})$ or to the north west vertex of the last tile in $i_j(\mathcal{G})$. Moreover, we require that i_1 and i_2 are maximal in the following sense:

(1) If $\mathcal{G}$ has at least two tiles and if there exists a snake graph $\mathcal{G}'$ with two embeddings $i'_1 : \mathcal{G}' \to \mathcal{G}_1$, $i'_2 : \mathcal{G}' \to \mathcal{G}_2$ such that $i_1(\mathcal{G}) \subseteq i'_1(\mathcal{G}')$ and $i_2(\mathcal{G}) \subseteq i'_2(\mathcal{G}')$ then $i_1(\mathcal{G}) = i'_1(\mathcal{G}')$ and $i_2(\mathcal{G}) = i'_2(\mathcal{G}')$.
(2) $\mathcal{G}$ is a snake graph consisting of at least one tile.

If the above the hold then we say that $\mathcal{G}_1$ and $\mathcal{G}_2$ have an *overlap* at $\mathcal{G}$.

In the case of a self-overlap we might have $i_1(\mathcal{G}) \cap i_2(\mathcal{G}) \neq \emptyset$.

Two snake graphs might have several overlaps with respect to the same and different snake sub-graphs.

Definition 2.10. [8, Definition 2.4] Let $\mathcal{G}_1 = (G_1, \dots, G_d)$ and $\mathcal{G}_2 = (G'_1, \dots, G'_{d'})$ be two snake graphs with overlap $\mathcal{G}$ and embeddings $i_1(\mathcal{G}) = \mathcal{G}_1[s, t]$ and $i_2(\mathcal{G}) = \mathcal{G}_2[s', t']$. Let $(\sigma_1, \dots, \sigma_{d-1})$ (respectively $(\sigma'_1, \dots, \sigma'_{d'-1})$) be the interior edges of $\mathcal{G}_1$ (respectively $\mathcal{G}_2$) and let f be a sign function on $\mathcal{G}$. Then f induces sign functions f_1 on $\mathcal{G}_1$ and f_2 on $\mathcal{G}_2$. We say that $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *cross in* $\mathcal{G}$ if one of the following conditions hold.

(1) $f_1(\sigma_{s-1}) = -f_1(\sigma_t)$ if $s > 1$ and $t < d$ or $f_2(\sigma'_{s'-1}) = -f_2(\sigma'_{t'})$ if $s' > 1$ and $t' < d'$,
(2) $f_1(\sigma_t) = f_2(\sigma'_{s'-1})$ if $s = 1$, $t < d$, $s' > 1$, and $t' = d'$ or $f_1(\sigma_{s-1}) = f_2(\sigma'_{t'})$ if $s > 1$, $t = d$, $s' = 1$, and $t' < d'$.

We call such an overlap a *crossing*.

We have a similar definition of a self-crossing overlap.

Definition 2.11. [8, Definition 2.6] Let $\mathcal{G}_1 = (G_1, \dots, G_d)$ be a snake graph with self-overlap $\mathcal{G}$ and embeddings $i_1(\mathcal{G}) = \mathcal{G}_1[s, t]$ and $i_2(\mathcal{G}) = \mathcal{G}_1[s', t']$. Let $(\sigma_1, \dots, \sigma_{d-1})$

be the interior edges of $\mathcal{G}_1$ and let f be a sign function on $\mathcal{G}_1$. We say that $\mathcal{G}_1$ has a *self-crossing in* $\mathcal{G}$ if one of the following conditions hold.

(1) $f(\sigma_{s-1}) = -f(\sigma_t)$ or $f(\sigma_{s'-1}) = -f(\sigma_{t'})$ if $t' < d$,
(2) $f(\sigma_t) = f(\sigma_{s'-1})$.

We call such an overlap a *self-crossing overlap*.

We will now see that a crossing or self-crossing overlap corresponds to a crossing or a self-crossing of arcs.

Theorem 2.12. *[8, Theorem 6.1] Let γ_1, γ_2 be (generalised) arcs and $\mathcal{G}_1, \mathcal{G}_2$ their corresponding snake graphs.*

(1) *γ_1, γ_2 cross with a nonempty local overlap $(\tau_{i_s}, \cdots, \tau_{i_t}) = (\tau_{i'_{s'}}, \cdots, \tau_{i'_{t'}})$ if and only if $\mathcal{G}_1, \mathcal{G}_2$ cross in $\mathcal{G}_1[s,t] \cong \mathcal{G}_2[s',t']$.*
(2) *γ_1 crosses itself with a non-empty local overlap $(\tau_{i_s}, \cdots, \tau_{i_t}) = (\tau_{i'_{s'}}, \cdots, \tau_{i'_{t'}})$ if and only if $\mathcal{G}_1$ has a self-crossing overlap $\mathcal{G}_1[s,t] \cong \mathcal{G}_1[s',t']$.*

In the proof of Theorem 4.1 we consider self-crossings of an arc. However, the way we treat a self-crossing of an arc is by replacing the one arc by two copies of the same arc. We then smooth crossings of the 'two' arcs as opposed to [8] where self-crossings of the single arc are smoothed. This can be done because of the following more general statement.

Proposition 2.13. *Let $\mathcal{G}_\gamma$ be a snake graph with a self-overlap $\mathcal{G}$. Then $\mathcal{G}$ is a self-crossing overlap of $\mathcal{G}_\gamma$ if and only if $\mathcal{G}$ is a crossing overlap of two copies of $\mathcal{G}_\gamma$.*

We note that here we consider the two copies of $\mathcal{G}_\gamma$ as 'distinct' snake graphs.

Proof. Suppose f is a sign function on $\mathcal{G}_\gamma = (G_1, \ldots, G_s, \ldots, G_{s'}, \ldots, G_d)$ and that $\mathcal{G}_\gamma$ has a self-crossing overlap $\mathcal{G}$. Denote by $(\sigma_1, \ldots, \sigma_{d-1})$ the interior edges of $\mathcal{G}_\gamma$. Then by Definition 2.11 there exist two embeddings $i_1(\mathcal{G}) = (G_s, \ldots, G_t)$ and $i_2(\mathcal{G}) = (G_{s'}, \ldots, G_{t'})$ where $s < s'$ and $f(\sigma_t) = f(\sigma_{s'-1})$. Now consider two copies of $\mathcal{G}_\gamma$, denote them by $\mathcal{G}_\gamma$, $\mathcal{G}'_\gamma$. Then we can canonically embed $(G_s, \ldots, G_t)$ into $\mathcal{G}_\gamma$ and $(G_{s'}, \ldots, G_{t'})$ into $\mathcal{G}'_\gamma$. Then by [7] this is an overlap for $\mathcal{G}_\gamma$ and $\mathcal{G}'_\gamma$ and the sign conditions for a crossing in the overlap $\mathcal{G}$ is satisfied by the sign function f on $\mathcal{G}_\gamma$ and on $\mathcal{G}'_\gamma$.

The converse immediately follows from a similar argument. □

It is clear that reversing the roles of $\mathcal{G}_\gamma$ and $\mathcal{G}'_\gamma$ gives a similar result for the second crossing of $\mathcal{G}_\gamma$ and $\mathcal{G}'_\gamma$ and it immediately implies that the number of self-crossings of an arc γ with itself times two is equal to the number of crossings of two copies of γ.

The smoothing of a crossing of two arcs γ_1 and γ_2 such that the associated snake graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ cross in an overlap is called, in terms of snake graphs, the *resolution of the overlap* [8]. It immediately follows from the definitions above and below that given a self-crossing of an arc γ with associated snake graph $\mathcal{G}_\gamma$, the two corresponding crossings of two copies of $\mathcal{G}_\gamma$ coincide and give rise to the same resolution.

We now define four snake graphs $\mathcal{G}_3$, $\mathcal{G}_4$, $\mathcal{G}_5$ and $\mathcal{G}_6$. For $\mathcal{G}[i,j] = (G_i, \dots, G_j)$, define $\overline{\mathcal{G}}[j,i] = (G_j, \dots, G_i)$. In order to define $\mathcal{G}_5$ and $\mathcal{G}_6$, we introduce the notation $\mathcal{G}'_5 = \mathcal{G}_1[1, s-1] \cup \overline{\mathcal{G}}_2[s'-1, 1]$ where we glue the two subgraphs along the edge with weight σ_s and $\mathcal{G}'_6 = \overline{\mathcal{G}}_2[d', t'+1] \cup \mathcal{G}_1[t+1, d]$ where we glue the two subgraphs along the edge with weight σ_t.

Let f_5 be a sign function on $\mathcal{G}'_5$ and f_6 a sign function on $\mathcal{G}'_6$.

We then define four snake graphs as follows.

$\mathcal{G}_3 = \mathcal{G}_1[1,t] \cup \mathcal{G}_2[t'+1, d']$ where the gluing of the two subgraphs is induced by the embedding i_2 of $\mathcal{G}$ in $\mathcal{G}_2$;

$\mathcal{G}_4 = \mathcal{G}_2[1,t'] \cup \mathcal{G}_1[t+1, d]$ where the gluing of the two subgraphs is induced by the embedding i_1 of $\mathcal{G}$ in $\mathcal{G}_1$;

$$\mathcal{G}_5 = \begin{cases} \mathcal{G}'_5 & \text{if } s > 1,\ s' > 1\ ; \\ \mathcal{G}'_5 \setminus \mathrm{Succ}(\sigma) & \text{if } s' = 1 \text{ where } \sigma \text{ is the last edge in } \mathrm{Int}(\mathcal{G}'_5) \cup {}_{SW}\mathcal{G}'_5 \text{ such that} \\ & f_5(\sigma) = f_5(\sigma_{s-1}); \\ \mathcal{G}'_5 \setminus \mathrm{Pred}(\sigma) & \text{if } s = 1 \text{ where } \sigma \text{ is the first edge in } \mathrm{Int}(\mathcal{G}'_5) \cup {\mathcal{G}'_5}^{NE} \text{ such that} \\ & f_5(\sigma) = f_5(\sigma'_{s-1}); \end{cases}$$

$$\mathcal{G}_6 = \begin{cases} \mathcal{G}'_6 & \text{if } t < d,\ t' < d'; \\ \mathcal{G}'_6 \setminus \mathrm{Succ}(\sigma) & \text{if } t = d, \text{ where } \sigma \text{ is the last edge in } \mathrm{Int}(\mathcal{G}'_6) \cup {}_{SW}\mathcal{G}'_6 \text{ such that} \\ & f_6(\sigma) = f_6(\sigma'_{t'}); \\ \mathcal{G}'_6 \setminus \mathrm{Pred}(\sigma) & \text{if } t' = d', \text{ where } \sigma \text{ is the first edge in } \mathrm{Int}(\mathcal{G}'_6) \cup {\mathcal{G}'_6}^{NE} \text{ such that} \\ & f_6(\sigma) = f_6(\sigma_t). \end{cases}$$

In the above definition we write $\mathrm{Res}_{\mathcal{G}}(\mathcal{G}_1, \mathcal{G}_2) = (\mathcal{G}_3 \sqcup \mathcal{G}_4, \mathcal{G}_5 \sqcup \mathcal{G}_6)$ for the resolution of the crossing of $\mathcal{G}_1$ and $\mathcal{G}_2$.

Theorem 2.14. *[8, Theorem 6.2] Let γ_1, γ_2 be (generalised) arcs and $\mathcal{G}_1, \mathcal{G}_2$ their corresponding snake graphs. The snake graphs of the four arcs obtained by smoothing the crossing of γ_1 and γ_2 in the overlap are given by the resolution* $\mathrm{Res}_{\mathcal{G}}(\mathcal{G}_1, \mathcal{G}_2)$ *of the crossing of the snake graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ at the overlap $\mathcal{G}$.*

We now consider a particular crossing of two arcs γ_1 and γ_2 such that this crossing does not correspond to a crossing overlap in the associated snake graphs $\mathcal{G}_1$ and $\mathcal{G}_2$. This situation occurs exactly if the crossing of γ_1 and γ_2 occurs in what we call in section 3 *a crossing in an arrow or in a 3-cycle* in $(S, \mathcal{M}, T)$. In particular, one (or both) of the arcs will have at least one endpoint coinciding with a vertex in that triangle. Following [8] there are two cases to consider.

Let $\mathcal{G}_1 = (G_1, G_2, \dots, G_d)$ and $\mathcal{G}_2 = (G'_1, G'_2, \dots, G'_{d'})$ be two snake graphs such that $G_s \neq G'_1$ for some $1 \leq s \leq d$ and let f_1 be a sign function on $\mathcal{G}_1$. Let δ be the unique common edge in $\mathcal{G}_s^{NE}$ and ${}_{SW}\mathcal{G}'_1$ if it exists. Let f_2 be a sign function on $\mathcal{G}_2$ such that $f_2(\delta) = f_1(\delta)$. Then define four snake graphs as follows.

Case 1. Suppose $s = d$.

$\mathcal{G}_3 = \mathcal{G}_1 \cup \mathcal{G}_2$ where the two subgraphs are glued along the edge δ;

$\mathcal{G}_4 = \{\delta\}$;

$\mathcal{G}_5 = \mathcal{G}_1 \setminus \mathrm{Succ}(\sigma)$, where $\sigma \in_{SW} \mathcal{G}_1 \cup \mathrm{Int}(\mathcal{G}_1)$ is the last edge such that $f_1(\sigma) = f_1(\delta)$;

$\mathcal{G}_6 = \mathcal{G}_2 \setminus \mathrm{Pred}(\sigma)$, where $\sigma \in \mathrm{Int}(\mathcal{G}_2) \cup \mathcal{G}_2^{NE}$ is the first edge such that $f_2(\sigma) = f_2(\delta)$.

Case 2. Suppose that $s \neq d$.

$\mathcal{G}_3 = \mathcal{G}_1[1, s] \cup \mathcal{G}_2$, where the two subgraphs are glued along the edge δ;

$\mathcal{G}_4 = \mathcal{G}_1 \setminus \mathrm{Pred}(\sigma)$, where $\sigma \in \mathrm{Int}(\mathcal{G}_1[s+1, d]) \cup \mathcal{G}_1^{NE}$ is the first edge such that $f_1(\sigma) = f_1(\delta)$;

$\mathcal{G}_5 = \mathcal{G}_1 \setminus \mathrm{Succ}(\sigma)$, where $\sigma \in_{SW} \mathcal{G}_1 \cup \mathrm{Int}(\mathcal{G}_1[1, s])$ is the last edge such that $f_1(\sigma) = f_1(\delta)$;

$\mathcal{G}_6 = \overline{\mathcal{G}}_2[d', 1] \cup \mathcal{G}_1[s+1, d]$, where the two subgraphs are glued along the edge σ_s.

In the above definitions we write $\mathrm{Graft}_{s,\delta}(\mathcal{G}_1, \mathcal{G}_2) = (\mathcal{G}_3 \sqcup \mathcal{G}_4, \mathcal{G}_5 \sqcup \mathcal{G}_6)$ and we call it grafting of $\mathcal{G}_2$ on $\mathcal{G}_1$ at s.

Similarly, we consider a particular self-crossing of an arc γ such that this crossing does not correspond to a crossing in an overlap in the associated snake graph $\mathcal{G}_\gamma$. As in the case of two distinct arcs this situation occurs exactly if the self-crossing of γ does not have any overlap. It is immediate that in terms of snake graphs detecting a self-crossing corresponding to a self-crossing of γ in a triangle is equivalent to considering the arc as two distinct arcs and two distinct snake graphs and detecting a crossing of two arcs in a triangle in terms of the two snake graphs.

Theorem 2.15. *[8, Theorem 6.4] Let γ_1 and γ_2 be two arcs which cross in a triangle Δ with an empty overlap, and let $\mathcal{G}_1$ and $\mathcal{G}_2$ be the corresponding snake graphs. Assume the orientation of γ_2 is such that Δ is the first triangle γ_2 meets. Then the snake graphs of the four arcs obtained by smoothing the crossing of γ_1 and γ_2 in Δ are given by the resolution* $\mathrm{Graft}_{s,\delta}(\mathcal{G}_1, \mathcal{G}_2)$ *of the grafting of $\mathcal{G}_2$ on $\mathcal{G}_1$ at s, where $0 \leq s \leq d$ is such that $\Delta_s = \Delta$ and if $s = 0$ or $s = d$ then δ is the unique side of Δ that is not crossed by either γ_1 or γ_2.*

Given a triangulation of $(S, \mathcal{M})$, we now establish a one-to-one correspondence between the set of snake graphs with a sign function (excluding snake graphs consisting of a single edge) and the set of string modules of the associated Jacobian algebra $J(Q, W)$. Let $M(w_\gamma)$ be the string module corresponding to an arc γ and let $\mathcal{G}_\gamma$ be the associated snake graph. Then the arrows and their formal inverses uniquely define a sign function f_γ on $\mathcal{G}_\gamma$. Namely, let $w_\gamma = \varepsilon_1 \dots \varepsilon_{d-1}$ and let $(\sigma_1, \dots, \sigma_{d-1})$ be the interior edges of $\mathcal{G}_\gamma$. Define a sign function f_γ on $\mathcal{G}_\gamma$ by setting $f_\gamma(\sigma_i) = \phi(\varepsilon_i)$ where $\phi(\varepsilon_i) = +$ if $\varepsilon_i \in Q_1$ and $\phi(\varepsilon_i) = -$ if $\varepsilon_i^{-1} \in Q_1$, for $1 \leq i \leq d-1$.

Set $\mathcal{R} = \{ \ (\mathcal{G}_\gamma, f_\gamma) \mid \gamma$ an arc in $(S, \mathcal{M}, T)$ and such that γ is not in $T\}$. The following result is immediate.

Proposition 2.16. *There is a bijection between the set of strings $\mathcal{S}$ over $J(Q, W)$ and the set $\mathcal{R}$ given by the map that associates $(\mathcal{G}_\gamma, f_\gamma)$ to the string w_γ for every arc γ in $(S, \mathcal{M}, T)$, $\gamma \notin T$.*

Such a correspondence in the setting of a triangulation of the once-punctured torus has also been noted in [39].

3. Extensions for the Jacobian algebra

Let $J(Q, W)$ be the Jacobian algebra associated to a triangulation of a marked surface $(S, \mathcal{M})$ with all marked points in the boundary and such that every boundary component contains at least one marked point.

In this section we interpret the crossing of arcs in terms of the corresponding string modules over $J(Q, W)$. We do this by using the characterization of crossing arcs in terms of snake graphs introduced in Section 2.4 and the snake graph and string module correspondence given in Proposition 2.16.

3.1. Crossing string modules

Given two (not necessarily distinct) arcs in a surface $(S, \mathcal{M})$, recall that there are three types of configurations in which these arcs can cross, see Fig. 3.

Each of these crossings gives rise to a different structure of the corresponding string modules which leads to Definition 3.1.

We use the notation $\mathrm{Pred}(\alpha)$ for the substring preceding an arrow or an inverse arrow α in a string w and similarly we use the notation $\mathrm{Succ}(\alpha)$ for the substring succeeding an arrow or an inverse arrow α in a string w.

Definition 3.1. We say that a string module M crosses a string module N if there exist strings w_M and w_N such that $M \simeq M(w_M)$ and $N \simeq M(w_N)$ and if one of the following three conditions hold

(1) there exists a string $w \in \mathcal{S}$, possibly consisting of a single vertex only, and such that w_M and w_N do not both start at $s(w)$ or do not both end at $t(w)$ and if

$$w_M = \mathrm{Pred}(\alpha) \xrightarrow{\alpha} w \xleftarrow{\beta} \mathrm{Succ}(\beta) \quad \text{and} \quad w_N = \mathrm{Pred}(\varepsilon) \xleftarrow{\varepsilon} w \xrightarrow{\delta} \mathrm{Succ}(\delta)$$

where $\alpha, \beta, \varepsilon, \delta$ are arrows in Q_1. If α (resp. ε) doesn't exist then w_M (resp. w_N) starts with w and if β (resp. δ) doesn't exist then w_M (resp. w_M) ends with w;

(2) there exists an arrow α in Q_1 such that $w_M \xrightarrow{\alpha} w_N \in \mathcal{S}$;

(3) if there exists a 3-cycle $\begin{tikzcd} a \arrow[r, "\alpha"] & b \arrow[d, "\delta"] \\ & c \arrow[ul, "\beta"] \end{tikzcd}$ in Q with $\alpha\delta, \delta\beta, \beta\alpha \in I$ and such that α is in w_M and $s(w_N) = c$ and w_N does not start or end with δ or β nor their inverses.

Moreover, if (1) holds we say that M crosses N in *a module*, if (2) holds we say that M crosses N in *an arrow* and if (3) holds we say that M crosses N in *a 3-cycle*. Moreover, if $M = N$, we say M has *a self-crossing* in *a module, an arrow, and a 3-cycle, respectively.*

Furthermore, if in (1) the arc $\gamma_{M(w)}$ is a self-crossing arc, we say M_1 crosses M_2 in a *self-crossing module* $M(w)$, and if $M = N$, we say that M has a *self-crossing in a self-crossing module* $M(w)$. In accordance with the terminology of snake graphs, we say that the corresponding arcs γ_1 and γ_2 have a *crossing in a self-crossing overlap*. If the arc $\gamma_{M(w)}$ has no self-crossing then we say that the respective crossing is in a *non-self-crossing module* $M(w)$.

Remark 3.2.

(1) We note that there is a direction in the crossings of modules. Namely, for a fixed crossing, we say that M crosses N or N crosses M. We will see in Theorem 3.7 that for a particular crossing, M crossing N will give rise to an element in $\mathrm{Ext}^1_{\mathcal{J}}(M, N)$ if the crossing is in a module or in an arrow whereas there will be no element in $\mathrm{Ext}^1_{\mathcal{J}}(N, M)$ corresponding to this crossing.

(2) It is possible for a string module M to cross a string module N simultaneously in a module, in an arrow and in a 3-cycle. In Section 5 we give an example of all three types of crossings of modules as defined above.

(3) Crossings in modules as described in 3.1(1) have also been considered in [6,42]. It also immediately follows from [11] that there is a non-zero homomorphism from N to M in that case.

(4) It is possible for M to cross N several times in modules, arrows and 3-cycles. For an example, see Section 5.

(5) It is possible for M to cross N and for N to cross M, see Section 5 for an example.

Proposition 3.3. *Let M and N be two string modules over $J(Q, W)$ with corresponding arcs γ_M and γ_N in $(S, \mathcal{M})$. Then γ_M and γ_N cross if and only if M crosses N or N crosses M.*

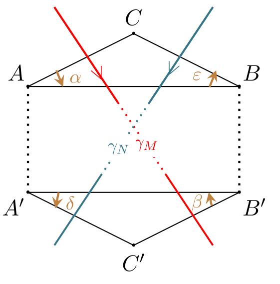


Fig. 4. Overlap crossing where both the arcs γ_M and γ_N might self-cross multiple times.

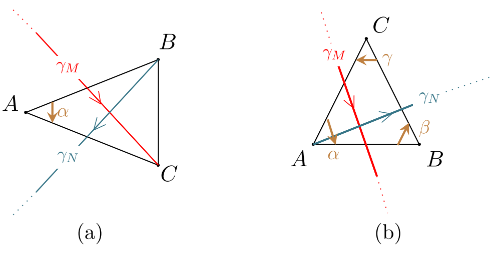


Fig. 5. Local configurations of crossings of γ_M and γ_N induced by the module M crossing the module N, corresponding in (a) to a crossing as in Definition 3.1 (2) and in (b) to Definition 3.1 (3).

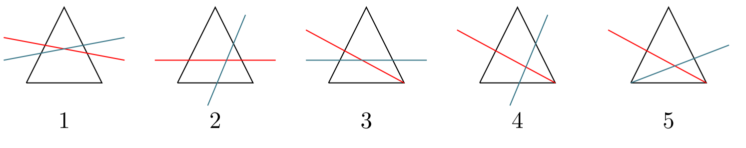


Fig. 6. The five possible ways two arcs can cross in any given triangle.

Proof. First assume, without loss of generality, that M crosses N, that is, Definition 3.1 (1), (2) or (3) holds. First suppose that M crosses N in a module. Let w_M and w_N be the associated strings as defined in 3.1 (1). Then if the arrows $\alpha, \beta, \delta, \varepsilon$ all exist, we have a local configuration as in Fig. 4.

The endpoints A, B, C, A', B' and C' in Fig. 4 might (all) coincide. If α does not exists then in the orientation given in Fig. 4, the arc γ_M starts at C and by definition the arrow ε must exist since, by definition, the strings w_M and w_N do not start at the same vertex. Similarly for the end points hence either δ or ε exists and thus γ_M and γ_N cross.

If M and N cross as in Definition 3.1 (2), then with the induced orientation on γ_M and γ_N, we have a local configuration as in Fig. 5 (a) and thus γ_M and γ_N cross.

If M and N cross as in Definition 3.1 (3), then with the induced orientation on γ_M, γ_N, we have a local configuration as in Fig. 5 (b) and γ_M and γ_N cross.

Conversely, suppose that γ_M and γ_N cross. Then there are 5 possible local configurations of a crossing of γ_M and γ_N as in Fig. 6.

By homotopy, configuration (1) can always be reduced to either configuration (2) or (3). So suppose that γ_M and γ_N are as in configuration (2) or (3) of Fig. 6. Then

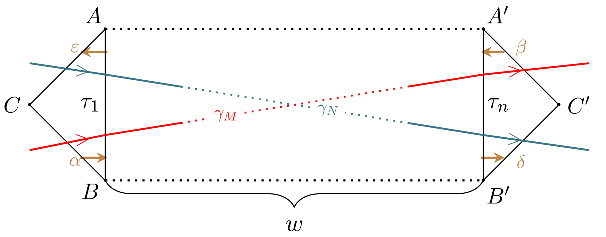


Fig. 7. Crossing of γ_M and γ_N where γ_M and γ_N both successively cross $\tau_1, \ldots, \tau_n$.

locally the crossing takes place in a configuration as in Fig. 7 where γ_M might start or end at C or C' and γ_N might start or end at C or C' but not both γ_M and γ_N start or end simultaneously at either point. Let w_M and w_N be the strings associated to the orientation of γ_M or γ_N in Fig. 7. We could also have chosen the inverse orientation for both γ_M and γ_N and then have worked with w_M^{-1} and w_N^{-1}. Given the chosen orientation, let γ_{AB} be the first arc of T both arcs γ_M and γ_N cross with respect to the crossing under consideration and suppose that γ_M and γ_N then both successively cross the arcs $\gamma_{AB} = \tau_1, \tau_2, \cdots, \tau_n$ of the triangulation T (where possibly $\tau_i = \tau_j$ for some i, j).

In order for γ_M and γ_N to cross as in Fig. 7 we must have either γ_M crossing τ_{CB} and γ_N crossing τ_{AC} or γ_N starting at C or if γ_M starts at C then γ_N must cross τ_{AC}.

If γ_M crosses τ_{CB} then $w_M = u_M \alpha w \beta^{-1} v_M$ and $w_N = u_N \varepsilon^{-1} w \delta v_N$ or $w_N = w \delta v_N$ for some strings w, u_M, u_N, v_M and v_N. If we set $\mathrm{Pred}(\alpha) = u_M$, $\mathrm{Pred}(\varepsilon) = u_N$, $\mathrm{Succ}(\beta) = v_M$ and $\mathrm{Succ}(\delta) = v_n$ then w_M and w_N are exactly as in Definition 3.1 (1) and thus M crosses N in a module.

If γ_M starts at C then $w_M = w\beta^{-1} v_M$ and $w_N = u_N \varepsilon^{-1} w \delta v_N$ for some strings w, u_M, u_N, v_M and v_N. Again the result follows if we set $\mathrm{Pred}(\alpha) = u_M$, $\mathrm{Pred}(\varepsilon) = u_N$, $\mathrm{Succ}(\beta) = v_M$ and $\mathrm{Succ}(\delta) = v_n$.

Similarly, if γ_M and γ_N end at D then the strings w_M and w_N are again exactly as in Definition 3.1 (1) and M crosses N in a module.

Suppose γ_M and γ_N cross locally as in configuration (4) in Fig. 6. Then there is an orientation of γ_M and γ_N such that we have a local configuration as in Fig. 5 (a).

Hence the crossing cannot be moved outside the triangle ABC by homotopy without increasing the number of intersections of γ_M or γ_N with the arcs of the triangulation. Therefore $e(w_M) = v_{\tau_{AB}}$ and $s(w_N) = v_{\tau_{AC}}$ and thus we can form the string $w_M \alpha w_N$ and M crosses N in an arrow.

Suppose γ_M and γ_N cross locally as in configuration (5) in Fig. 6 which corresponds to a crossing of oriented arcs as in Fig. 5 (b). Again the crossing cannot be moved outside the triangle ABC by homotopy without increasing the number of intersection of γ_M and γ_N with the arcs of the triangulation.

Then $\begin{tikzcd} a \arrow[r, "\alpha"] & b \arrow[d, "\gamma"] \\ & c \arrow[ul, "\sigma"] \end{tikzcd}$ is a 3-cycle such that $\alpha\gamma, \gamma\sigma, \sigma\alpha$ are in I. Furthermore, α is in w_M,

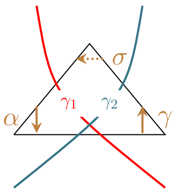


Fig. 8. Existence of the arrow σ in the case of a crossing in a module.

$s(w_N) = c$ and w_N does not start with α or γ nor their inverses. Hence M crosses N in a 3-cycle. $\square$

We now show the existence of some arrows that occur if M crosses N in a module. These arrows will be needed in Definition 3.5 (1) below.

Lemma 3.4. *Let Q be a quiver associated to a triangulation of $(S, \mathcal{M})$. Suppose two strings w and v are of the form $w = \mathrm{Pred}(\alpha) \xrightarrow{\alpha} u \xleftarrow{\beta} \mathrm{Succ}(\beta)$ and $v = \mathrm{Pred}(\gamma) \xleftarrow{\gamma} u \xrightarrow{\delta} \mathrm{Succ}(\delta)$ where $u \in \mathcal{S}$ and $\alpha, \beta, \gamma, \delta$ are arrows in Q_1. Then if the arrows α and γ exist with $\alpha\gamma \in I$ then there exists an arrow σ in Q such that* $\begin{tikzcd} a \arrow[r, "\alpha"] & b \arrow[d, "\gamma"] \\ & c \arrow[ul, "\sigma"] \end{tikzcd}$ *is a 3-cycle in Q and $\gamma\sigma \in I$ and $\sigma\alpha \in I$ and $b = s(u)$.*

Similarly, if the arrows β and δ exist with $\beta\delta \in I$ then there exists an arrow ρ in Q such that $\begin{tikzcd} d \arrow[dr, "\delta"] & e \arrow[l, "\beta"'] \\ & f \arrow[u, "\rho"] \end{tikzcd}$ *is a 3-cycle in Q and $\beta\rho \in I$ and $\rho\delta \in I$ and $d = t(u)$.*

Proof. Let $u = u_1 \overset{\mu_1}{\text{——}} u_2 \overset{\mu_2}{\text{——}} u_3 \cdots u_r \overset{\mu_r}{\text{——}} u_{r+1}$. Then $\alpha\mu_1 \in \mathcal{S}$ and $\gamma^{-1}\mu_1 \in \mathcal{S}$ and $\alpha\gamma$ is a non-zero path in Q. Since either $\mu_1 \in Q_1$ or $\mu_1^{-1} \in Q_1$ we have either $\alpha\mu_1 \in kQ$ and $\alpha\mu_1 \notin I$ or $\mu_1^{-1}\gamma \in kQ$ and $\mu_1^{-1}\gamma \notin I$. Since $J(Q, W)$ is gentle, by (S2) we have $\alpha\gamma \in I$. Since $J(Q, W)$ is a gentle algebra coming from a surface triangulation this implies that there exists an arrow σ such that $\alpha\gamma\sigma$ is a 3-cycle in Q and that $\gamma\sigma \in I$ and $\sigma\alpha \in I$, see Fig. 8.

A similar argument proves the existence of the 3-cycle containing ρ. $\square$

3.2. Smoothing of crossings for string modules

Given two arcs in $(S, \mathcal{M})$ that cross, the smoothing of a crossing gives rise to four new arcs as in Fig. 9. We interpret these arcs in terms of string modules.

Given two string modules $M_1 = M(w_1)$ and $M_2 = M(w_2)$ in mod $J(Q, W)$ such that the associated arcs γ_1 and γ_2 cross in $(S, \mathcal{M})$, we define four new modules $M_3 = M(w_3)$,

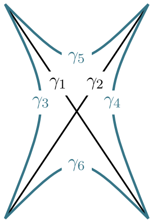


Fig. 9. Smoothing a crossing of γ_1 and γ_2.

$M_4 = M(w_4)$, $M_5 = M(w_5)$ and $M_6 = M(w_6)$. By possibly relabelling M_1 and M_2 we can always assume that M_1 crosses M_2.

Definition 3.5. Let M_1 and M_2 be two string modules over $J(Q,W)$ with strings w_1 and w_2, respectively.

(1) Suppose M_1 crosses M_2 in a module $M(w)$. That is

$$w_1 = \mathrm{Pred}(\alpha) \xrightarrow{\alpha} w \xleftarrow{\beta} \mathrm{Succ}(\beta) \quad \text{and} \quad w_2 = \mathrm{Pred}(\gamma) \xleftarrow{\gamma} w \xrightarrow{\delta} \mathrm{Succ}(\delta)$$

where $\alpha, \beta, \gamma, \delta$ are arrows in Q_1. Define

$$w_3 = \mathrm{Pred}(\alpha) \xrightarrow{\alpha} w \xrightarrow{\delta} \mathrm{Succ}(\delta)$$
$$w_4 = \mathrm{Pred}(\gamma) \xleftarrow{\gamma} w \xleftarrow{\beta} \mathrm{Succ}(\beta)$$
$$w_5 = \begin{cases} \mathrm{Pred}(\alpha) \xleftarrow{\sigma} \mathrm{Pred}^{-1}(\gamma) & \text{if } s(w_1) \neq s(w) \text{ and } s(w_2) \neq s(w) \\ \mathrm{Pred}(\alpha)_c & \text{if } s(w_2) = s(w) \\ \mathrm{Pred}(\gamma)_h & \text{if } s(w_1) = s(w) \end{cases}$$
$$w_6 = \begin{cases} \mathrm{Succ}^{-1}(\beta) \xleftarrow{\rho} \mathrm{Succ}(\delta) & \text{if } t(w_1) \neq t(w) \text{ and } t(w_2) \neq t(w) \\ {}_c\mathrm{Succ}(\beta) & \text{if } t(w_2) = t(w) \\ {}_h\mathrm{Succ}(\delta) & \text{if } t(w_1) = t(w) \end{cases}$$

where $\sigma, \rho \in Q_1$ are as in Lemma 3.4.

(2) Suppose M_1 crosses M_2 in an arrow α. Suppose without loss of generality that $s(\alpha) = t(w_1)$ and $t(\alpha) = s(w_2)$. Define

$$w_3 = w_1 \xrightarrow{\alpha} w_2$$
$$w_4 = 0$$
$$w_5 = (w_1)_c$$
$$w_6 = {}_h w_2.$$

(3) Suppose M_1 crosses M_2 in the 3-cycle

$$\begin{tikzcd} s(w_2) \arrow[r, "\gamma"] & s(\alpha) \arrow[d, "\alpha"] \\ & t(\alpha) \arrow[ul, "\beta"] \end{tikzcd}$$

where (modulo inverting w_1) $w_1 = \mathrm{Pred}(\alpha) \xrightarrow{\alpha} \mathrm{Succ}(\alpha)$. Define

$$w_3 = \mathrm{Pred}(\alpha) \xleftarrow{\gamma} w_2$$
$$w_4 = {}_c(s(\alpha) \xrightarrow{\alpha} \mathrm{Succ}(\alpha))$$

$$w_5 = (\mathrm{Pred}(\alpha) \xrightarrow{\alpha} t(\alpha))_h$$
$$w_6 = w_2^{-1} \xleftarrow{\beta} \mathrm{Succ}(\alpha).$$

Just as is the case for snake graphs, if $M_1 = M_2$ then one self-crossing gives rise to two crossings of modules when we consider two copies of the same module. However, the smoothing of either of the two crossings results in the same four modules. Therefore it is enough to consider only one of the crossings in each case.

Proposition 3.6. *Let γ_1 and γ_2 be two crossing arcs in (S, M, T) with strings w_1 and w_2 respectively. Consider a given crossing of γ_1 and γ_2. Then the arcs $\gamma_3, \gamma_4, \gamma_5$ and γ_6 defined by the strings in Definition 3.5 correspond to the arcs obtained by smoothing the given crossing of γ_1 and γ_2 as in Definition 2.4.*

Proof. This follows directly from the resolution and grafting of snake graphs in [7] which for the convenience of the reader we have recalled in Section 2.4 in Theorems 2.14 and 2.15. □

We show that this enables us for any crossing to characterise whether it gives rise to a short exact sequence in $J(Q, W)$ or not and if there is a short exact sequence we describe the middle terms in terms of the arcs obtained from smoothing the crossing.

Theorem 3.7. *Let M_1 and M_2 be string modules in* mod $J(Q, W)$.

(1) *If M_1 crosses M_2 in a module then the modules M_3 and M_4 defined in Definition 3.5(1) above give a non-split short exact sequence in* mod $J(Q, W)$ *of the form*

$$0 \longrightarrow M_2 \longrightarrow M_3 \oplus M_4 \longrightarrow M_1 \longrightarrow 0.$$

(2) *If M_1 crosses M_2 in an arrow then the module M_3 defined in Definition 3.5(2) above gives a non-split short exact sequence in* mod $J(Q, W)$ *of the form*

$$0 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow M_1 \longrightarrow 0.$$

(3) *If M_1 crosses M_2 in a 3-cycle then the modules M_3 and M_4 defined in Definition 3.5(3) above do not give rise to an element in* $\mathrm{Ext}^1_{J(Q,W)}(M_1, M_2)$.

Proof. We use the notation of Definition 3.5.

(1) Set $w_1 = P_1 w S_1$ where $P_1 = \mathrm{Pred}(\alpha) \xrightarrow{\alpha}$ and $S_1 = \xleftarrow{\beta} \mathrm{Succ}(\beta)$ and set $w_2 = P_2 w S_2$ where $P_2 = \mathrm{Pred}(\gamma) \xleftarrow{\gamma}$ and $S_2 = \xrightarrow{\delta} \mathrm{Succ}(\delta)$. By definition we do not simultaneously have $P_1 = 0$ and $P_2 = 0$ or $S_1 = 0$ and $S_2 = 0$. Thus by [40] the sequence $0 \longrightarrow M_2 \longrightarrow M_3 \oplus M_4 \longrightarrow M_1 \longrightarrow 0$ is a non-split short exact sequence.

(2) This follows directly from the canonical embedding $M(w_2) \longrightarrow M(w_1 \xrightarrow{\alpha} w_2)$ and the canonical projection $M(w_1 \xrightarrow{\alpha} w_2) \longrightarrow M(w_1)$, see [5].

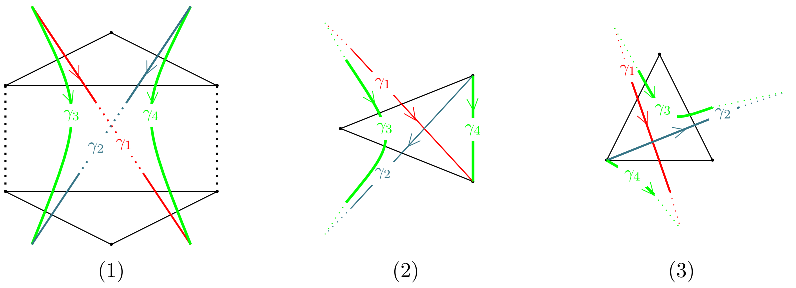


Fig. 10. Geometric interpretation of Theorem 3.7.

(3) Since $\dim M({}_c(\xrightarrow{\alpha} \mathrm{Succ}(\alpha)) < \dim M(\xrightarrow{\alpha} \mathrm{Succ}(\alpha))$ it is immediate by comparing the dimensions of $M_1 \oplus M_2$ and $M_3 \oplus M_4$ that these four modules cannot give rise to a short exact sequence in mod $J(Q, W)$. $\square$

Remark 3.8.

(1) The geometric interpretation of the module crossings in Theorem 3.7 is as follows: For Theorem 3.7 (1), the module M_1 crossing the module M_2 corresponds to a crossing of the corresponding arcs γ_1 and γ_2 as in Fig. 10 (1) and the modules M_3 and M_4 correspond to the arcs γ_3 and γ_4.
If M_1 crosses M_2 as in Theorem 3.7 (2), in the geometric set-up this corresponds to a crossing of arcs as in Fig. 10 (2). In this case γ_4 is either a boundary arc or an arc in the triangulation and the corresponding module M_4 is the zero module.
For modules crossings as in Theorem 3.7 (3), the geometric picture is as in Fig. 10 (3).

(2) In general, M_5 and M_6 as defined above never give rise to an element in $\mathrm{Ext}^1_{J(Q,W)}(M_1, M_2)$ nor in $\mathrm{Ext}^1_{J(Q,W)}(M_2, M_1)$ since $\dim(M_5 \oplus M_6) \leq \dim(M_1 \oplus M_2) - 1$.

(3) If $M_2 = M(\gamma)$ and $M_1 = M(s\gamma_e)$ for an arc γ in $(S, \mathcal{M})$ where ${}_s\gamma_e$ is the arc rotated by the elementary pivot moves as defined in [6] then in Theorem 3.7(i) and (ii) above we recover the AR-sequences described in [6].

4. Extensions in the cluster category and dimension formula for Jacobian algebras

In this section let $\mathcal{C}(S, \mathcal{M})$ be the cluster category of a marked surface $(S, \mathcal{M})$ where all marked points lie in the boundary of S and each boundary component has at least one marked point.

In Theorem 4.1 we explicitly describe the middle terms of triangles in $\mathcal{C}(S, \mathcal{M})$. Namely, we show that any crossing of two arcs gives rise to at least one triangle in the cluster category where the middle terms of the triangle are given by one of the pairs of arcs obtained by smoothing the crossing. In almost all cases, except if the crossing has a self-overlap as defined in Definition 3.1, the middle terms of the triangle in the opposite direction are given by the other pair of arcs obtained by smoothing the crossing.

Let $M_1 = M(\gamma_1)$ and $M_2 = M(\gamma_2)$ be two string $J(Q,W)$-modules corresponding to the string objects γ_1 and γ_2 in $\mathcal{C}(S,\mathcal{M})$. It follows immediately from [35] (see also [23, Lemma 4.4]) that every short exact sequence in mod $J(Q,W)$ lifts to a triangle in $\mathcal{C}(S,\mathcal{M})$ such that the image under the canonical projection functor from $\mathcal{C}(S,\mathcal{M})$ to mod $J(Q,W)$ is isomorphic to the short exact sequence. Since both the indecomposable objects in the cluster category and as well as the indecomposable modules over the Jacobian algebra correspond to arcs in the surface (where the canonicity of this bijection follows from the Appendix to this paper), we can use the associated string to explicitly check whether we have a short exact sequence in the module category. It then follows that this short exact sequence comes from the triangle corresponding to the same set of arcs in the associated cluster category. Therefore by Theorem 3.7, we have that

(*) if M_1 crosses M_2 in a module and if $M_3 = M(\gamma_3)$ and $M_4 = M(\gamma_4)$ are defined as in Definition 3.5(1) then by there is a non-split triangle in $\mathcal{C}(S,\mathcal{M})$ given by

$$\gamma_2 \longrightarrow \gamma_3 \oplus \gamma_4 \longrightarrow \gamma_1 \longrightarrow \gamma_2[1].$$

Note that in this situation, even if the module crossing is self-crossing, we obtain this triangle in $\mathcal{C}(S,\mathcal{M})$.
(**) if M_1 crosses M_2 in an arrow α and if $M_3 = M(\gamma_3)$ is defined as in Definition 3.5(2) then there is a non-split triangle in $\mathcal{C}(S,\mathcal{M})$ given by

$$\gamma_2 \longrightarrow \gamma_3 \oplus \gamma_4 \longrightarrow \gamma_1 \longrightarrow \gamma_2[1]$$

where $\gamma_4 \neq 0$ if and only if γ_4 is not a boundary arc.

Theorem 4.1. *Let γ_1 and γ_2 be two string objects (not necessarily distinct) in $\mathcal{C}(S,\mathcal{M})$ such that their corresponding arcs cross in $(S,\mathcal{M})$. Let $\gamma_3, \gamma_4, \gamma_5, \gamma_6$ be the string objects corresponding to the smoothing of a crossing of a suitable orientation of the corresponding arcs γ_1 and γ_2. Then there is a non-split triangle in $\mathcal{C}(S,\mathcal{M})$ given by*

$$\gamma_2 \longrightarrow \gamma_3 \oplus \gamma_4 \longrightarrow \gamma_1 \longrightarrow \gamma_2[1] \tag{3}$$

and if the crossing of γ_1 and γ_2 is not in a self-crossing overlap in some triangulation of $(S,\mathcal{M})$ then we obtain a non-split triangle given by

$$\gamma_1 \longrightarrow \gamma_5 \oplus \gamma_6 \longrightarrow \gamma_2 \longrightarrow \gamma_1[1] \tag{4}$$

where $\gamma_3, \gamma_4, \gamma_5, \gamma_6$ are zero objects in $\mathcal{C}(S,\mathcal{M})$ if they correspond to boundary arcs.

Before we give a proof of Theorem 4.1 in Section 4.1, we explore some consequences.

Remark 4.2. Keeping the notations of Theorem 4.1, suppose that we are given a particular triangulation of $(S,\mathcal{M})$. Then we have the following two facts:

(i) For every crossing of γ_1 and γ_2 that corresponds to either an arrow crossing or a 3-cycle crossing of the associated string modules, we always obtain exactly two triangles in the cluster category with middle terms given by $\gamma_3 \oplus \gamma_4$ and by $\gamma_5 \oplus \gamma_6$, respectively.

(ii) For every crossing of γ_1 and γ_2 that corresponds to a non-selfcrossing module crossing of the associated string modules we obtain exactly two triangles in the cluster category with middle terms given by $\gamma_3 \oplus \gamma_4$ and by $\gamma_5 \oplus \gamma_6$. If the module crossing is self-crossing, we only obtain one of the two triangles in the cluster category and the middle terms of that triangle are given by one of the two pairs of arcs obtained from smoothing the crossing. In Theorem 4.1 we have denoted this pair of arcs by $\gamma_3 \oplus \gamma_4$. By the 2-Calabi Yau property of the cluster category and by [42] we know that this crossing must also give rise to a second triangle.

Question 4.3. *It is an open question whether, in the self-crossing module crossing case, the middle terms of the second triangle in Remark 4.2 (ii) are also induced by the 'other' pair of arcs, which in the notation of Theorem 4.1 are γ_5 and γ_6.*

Combining Theorem 3.7 with Theorem 4.1 and [42, Theorem 3.4] we obtain a formula for the dimensions of the first extension space over the Jacobian algebra in the case where the arcs do not create a self-crossing overlap.

Corollary 4.4. *Let M, N be two string modules over $J(Q,W)$ and let γ_M and γ_N be the corresponding arcs in $(S, \mathcal{M})$ such that γ_M and γ_N have no crossing with self-crossing overlap.*

(1) *A basis of $\operatorname{Ext}^1_{J(Q,W)}(M,N)$ is given by all short exact sequences arising from M crossing N in a module or an arrow and where the middle terms are as described in Theorem 3.7;*

(2) *We have*

$$\dim \operatorname{Ext}^1_{J(Q,W)}(M,N) + \dim \operatorname{Ext}^1_{J(Q,W)}(N,M) = \operatorname{Int}(\gamma_M, \gamma_N) - k - k'$$

where k (resp. k') is the number of times that M crosses N (resp. N crosses M) in a 3-cycle. In particular, if $M = N$ we have

$$2 \dim \operatorname{Ext}^1_{J(Q,W)}(M,M) = \operatorname{Int}(\gamma_M, \gamma_M) - 2k.$$

Proof. (1) Fix a crossing p of γ_M and γ_N and suppose without loss of generality that this crossing yields M crosses N in mod $J(Q,W)$. Then M crosses N either in a non-self-crossing module, in an arrow, or in a 3-cycle. Let $\gamma_3, \ldots, \gamma_6$ be the arcs obtained by smoothing the crossing p. By view of Theorem 4.1, it is sufficient to check whether $M(\gamma_3), M(\gamma_4)$ or $M(\gamma_5), M(\gamma_6)$ appear as middle terms of a non-split short exact sequence from N to M. If M crosses N in a non-self-crossing module or in an arrow at p, then by Theorem 3.7 there exists a non-split short exact sequence with middle terms given by $M(\gamma_3)$ and $M(\gamma_4)$. If M crosses N in a 3-cycle, then by Theorem 3.7 this does not result in a short exact sequence from N to M with middle terms $M(\gamma_3)$ and $M(\gamma_4)$. The modules $M(\gamma_5)$ and $M(\gamma_6)$ never induce an element in $\mathrm{Ext}^1_{J(Q,W)}(M,N)$.

(2) From the proof of (1) we see that a dimension formula for $\mathrm{Ext}^1_{J(Q,W)}(M,N)$ accounts only for the number of times M crosses N in a (non-self-crossing) module or in an arrow. The result then follows from the dimension formula in [42] and Theorem 3.7. $\square$

4.1. *Proof of Theorem 4.1*

Our general strategy for the proof of Theorem 4.1 is as follows.

We consider each type of crossing separately. That is, given a fixed triangulation T of $(S,\mathcal{M})$ and two string objects γ_1 and γ_2 in $\mathcal{C}(S,\mathcal{M})$ corresponding to two crossing arcs in $(S,\mathcal{M})$, we treat the different crossings of the corresponding string modules $M_1 = M(w_1)$ and $M_2 = M(w_2)$ one by one.

If the crossing under consideration is a crossing in a module then by (*) above we obtain one triangle with two middle terms given by the string objects γ_3 and γ_4. The other triangle is obtained by possibly flipping the overlap to an *orthogonal overlap* (see proof of Theorem 4.1, Section 4.1.1, *Case 1* below for the definition. However, sometimes this is not possible. In this case we adapt a strategy similar to the one in [42]. That is, we increase the number of marked points in the surface by one or two points to obtain a surface $(S,\mathcal{M}')$ where $\mathcal{M} \subset \mathcal{M}'$. We triangulate $(S,\mathcal{M}')$ by adding one or two arcs and flip the—now bigger—overlap to an orthogonal overlap. This gives rise to a triangle from γ_1 to γ_2 with middle terms γ_5 and γ_6 in $\mathcal{C}(S,\mathcal{M}')$ where here γ_1 and γ_2 are considered as arcs in $(S,\mathcal{M}')$. Then by flipping the new (orthogonal) arc and using the cutting procedure described in [30] which is compatible with Iyama–Yoshino reduction [22], we obtain the triangle involving the arcs γ_5 and γ_6 in $\mathcal{C}(S,\mathcal{M})$.

If the crossing under consideration is an arrow crossing then by (**) above we obtain a triangle with middle terms corresponding to the arcs γ_3 and γ_4. The other triangle is then obtained by either flipping an arc in the triangulation and thus creating an overlap (i.e. a module crossing) which we can flip to an orthogonal overlap or, if this is not possible, by adding a marked point to obtain a surface $(S,\mathcal{M}')$ with $\mathcal{M} \subset \mathcal{M}'$ and completing it to a triangulation of $(S,\mathcal{M}')$. In which case we obtain an overlap which we can flip to an orthogonal overlap. This gives rise to a triangle in $\mathcal{C}(S,\mathcal{M}')$. By cutting according to [30], we obtain the corresponding triangle from γ_1 to γ_2 in $\mathcal{C}(S,\mathcal{M})$ with middle terms γ_5 and γ_6.

If the crossing is a crossing in a 3-cycle then none of the triangles in $\mathcal{C}(S,\mathcal{M})$ are obtained from non-split short exact sequences in the Jacobian algebra corresponding to the given triangulation T. Instead we change the triangulation to create a crossing in a module—by possibly adding a marked point. Once we are in the case of a module crossing we can adapt the described strategy for module crossings above.

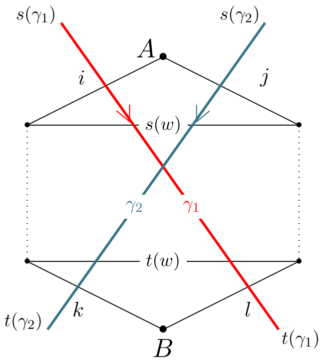


Fig. 11. *Case 1*: Local configuration of γ_1 and γ_2 crossing in a module where A and B might coincide.

4.1.1. Module crossing

Here we consider the case that $M_1 = M(w_1)$ crosses $M_2 = M(w_2)$ in a module. In terms of snake graphs a crossing in a module corresponds to a crossing in an overlap $\mathcal{G}$. Therefore as explained above, by Theorem 3.7 there always is a non-split triangle in $\mathcal{C}(S, \mathcal{M})$ given by

$$\gamma_2 \longrightarrow \gamma_3 \oplus \gamma_4 \longrightarrow \gamma_1 \longrightarrow \gamma_2[1].$$

When we give an orientation to an arc γ, we call $s(\gamma)$, the marked point at which γ starts and $t(\gamma)$ the marked point at which γ ends. In order to prove the existence of the triangle involving γ_5 and γ_6, there are several cases to consider depending on where the arcs γ_1 and γ_2 start and end with respect to the overlap.

Let $w_1 = P_1 w S_1$ and $w_2 = P_2 w S_2$ where w corresponds to the overlap $\mathcal{G}$. Let $\tau_1, \tau_2, \ldots, \tau_n$ be the arcs corresponding to the overlap $\mathcal{G}$, that is $s(w) = \tau_1$ and $t(w) = \tau_n$.

Case 1: $P_1 \neq 0$, $S_1 \neq 0$, $P_2 \neq 0$, $S_2 \neq 0$.

In $(S, \mathcal{M})$ this corresponds to the local configuration as in Fig. 11.

In particular, M_1 crosses M_2 such that $t(P_1) = i$ and $t(P_2) = j$ and $s(S_1) = l$ and $s(S_2) = k$.

Since the crossing is not a self-crossing overlap, there exists an arc τ_{AB} from A to B crossing the arcs $\tau_1, \ldots, \tau_n$. Let T' be the triangulation containing τ_{AB} and such that the flips of $\tau_1, \ldots, \tau_n$ in T' are connected to A and do not cross the arc τ_{AB}. Locally τ_{AB} lies in T' as in Fig. 12.

We denote by $J(Q', W')$ the Jacobian algebra with respect to the new triangulation T'. Let M_1' and M_2' be the string modules over $J(Q', W')$ corresponding to the arcs γ_1 and γ_2. Now M_2' crosses M_1' in a new overlap corresponding to τ_{AB}. We call this new overlap an *orthogonal flip* of the overlap $\mathcal{G}$.

More explicitly, let w_1' (resp. w_2') be the string of γ_1 (resp. γ_2) in $(S, \mathcal{M}, T')$. Then w_1' contains the subword $b \longleftarrow a \longrightarrow d$ and $(w_2')^{-1}$ contains the subword $e \longrightarrow a \longleftarrow c$

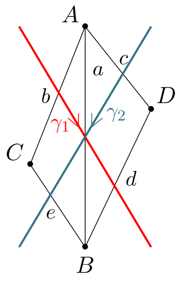


Fig. 12. Triangulation T' where a denotes the arc τ_{AB}.

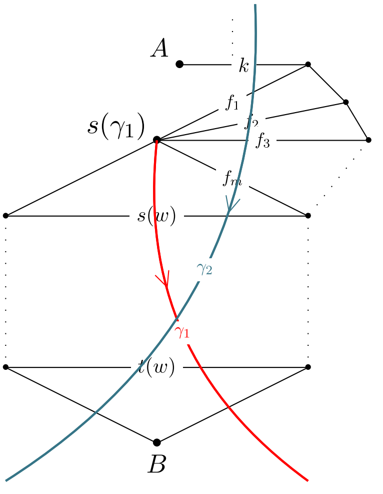


Fig. 13. *Case 2 (i) a):* Local configuration of γ_1 and γ_2 where A, $s(\gamma_1)$ and $s(\gamma_2)$ may coincide.

where, as in Fig. 12, a denotes the arc τ_{AB}. This gives rise to $M_2' = M(w_2')$ crossing $M_1' = M(w_1')$ in the simple module $M(a)$.

By Theorem 3.7(i) this gives rise to a non-split short exact sequence

$$0 \longrightarrow M_1' \longrightarrow M_3' \oplus M_4' \longrightarrow M_2' \longrightarrow 0$$

where M_3' (resp. M_4') is the string module over $J(Q', W')$ corresponding to the arc γ_5 (resp. γ_6) with respect to T'. Thus in $\mathcal{C}(S, \mathcal{M})$ there is a triangle

$$\gamma_1 \longrightarrow \gamma_5 \oplus \gamma_6 \longrightarrow \gamma_2 \longrightarrow \gamma_1[1].$$

Case 2 (i): $P_1 = 0$, $S_1 \neq 0$, $P_2 \neq 0$, $S_2 \neq 0$.

(a) Suppose P_2 is not a direct string. In this case we have the local configuration as in Fig. 13.

Since P_2 is not a direct string, there must exist a marked point A and an arc k and a maximal fan $f_1, \ldots, f_n$ such that γ_2 crosses k and $f_1, \ldots, f_n$, as in Fig. 13. Let B be the

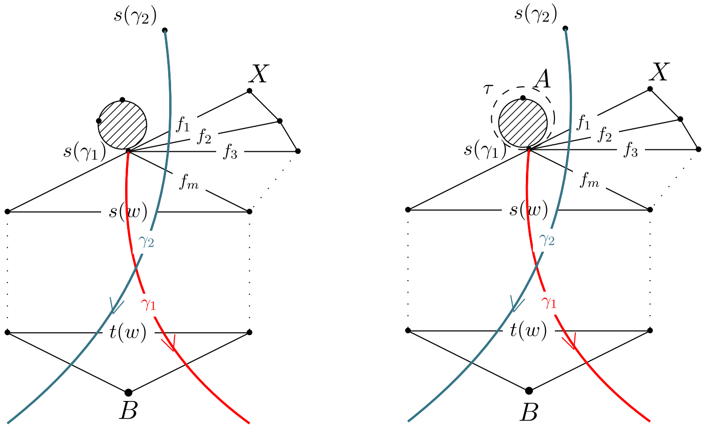


Fig. 14. *Case 2 (i) b):* Left figure: local configuration in $(S, \mathcal{M}, T)$ where the boundary component has several marked points. Right figure: local configuration in $(S, \mathcal{M}, T)$ where $s(\gamma_1)$ is the only marked point on the boundary component.

marked point in the triangle $Bs(\tau_n)t(\tau_n)$ where γ_2 crosses the arc from $s(\tau_n)$ to B, as in Fig. 13. Then there exists a triangulation T' of $(S, \mathcal{M})$ containing the arc τ_{AB} starting at A, ending at B, crossing both γ_1 and γ_2 and such that locally no other arc in T' crosses both γ_1 and γ_2.

Therefore by Theorem 3.7(i) we obtain a non-split short exact sequence

$$0 \longrightarrow M_1' \longrightarrow M_3' \oplus M_4' \longrightarrow M_2' \longrightarrow 0$$

where M_3' (resp. M_4') is the string module over $J(Q', W')$ corresponding to the arc γ_5 (resp. γ_6) with respect to T'. Thus in $\mathcal{C}(S, \mathcal{M})$ there is a triangle

$$\gamma_1 \longrightarrow \gamma_5 \oplus \gamma_6 \longrightarrow \gamma_2 \longrightarrow \gamma_1[1].$$

(b) Suppose now that P_2 is a direct string, see Fig. 14.

Our argument is based on the boundary component $\mathcal{B}$ containing $s(\gamma_1)$.

(I) Suppose that $\mathcal{B}$ contains another marked point A which is not equal to $s(\gamma_1)$ and $s(\gamma_2)$ is not in $\mathcal{B}$. Then there is an arc τ_{AB} from A to B crossing γ_1 and γ_2, see left hand side of Fig. 14 and we conclude as in part (a).
(II) Suppose $s(\gamma_1)$ is the only marked point on $\mathcal{B}$ and that $s(\gamma_1) \neq s(\gamma_2)$. Then consider instead the surface $(S, \mathcal{M}')$ where $\mathcal{M}' \supset \mathcal{M}$ has exactly one more marked point A than $\mathcal{M}$ lying on the boundary component $\mathcal{B}$, see right hand side of Fig. 14. Complete T to a triangulation T' on $(S, \mathcal{M}')$ by adding one new arc τ from A to X, where X is as in Fig. 14. Therefore the same method as in part (a) can be applied and we obtain a triangle in $\mathcal{C}(S, \mathcal{M}')$

$$\gamma_1 \longrightarrow \gamma_5 \oplus \gamma_6 \longrightarrow \gamma_2 \longrightarrow \gamma_1[1].$$

Note that since $\mathcal{M} \subset \mathcal{M}'$, whenever we have an arc in $(S, \mathcal{M}')$ between marked points $a, b \in \mathcal{M}'$ such that $a, b \in \mathcal{M}$, by a slight abuse of notation we use the same notation for this arc as an arc in $(S, \mathcal{M})$ and as an arc in $(S, \mathcal{M}')$.

Now flip the triangulation T' to a triangulation T'' such that T'' contains the arc τ' around the boundary component $\mathcal{B}$ from $s(\gamma_1)$ to $s(\gamma_1)$, see right hand side of Fig. 14. Then there is a triangle $s(\gamma_1)s(\gamma_1)A$ in T''. Cutting τ' as defined in [30] gives a surface isotopic to $(S, \mathcal{M})$ since we delete any component homeomorphic to a triangle after the cut.

Note that by [30] the arcs in $(S, \mathcal{M}')$ not crossing τ' are in bijection with the arcs in $(S, \mathcal{M})$. Since τ' is a boundary segment in $(S, \mathcal{M})$ the corresponding object in $\mathcal{C}(S, \mathcal{M})$ is the zero object. Thus by Proposition 5 in [30] we obtain a triangle in $\mathcal{C}(S, \mathcal{M})$

$$\gamma_1 \longrightarrow \gamma_5 \oplus \gamma_6 \longrightarrow \gamma_2 \longrightarrow \gamma_1[1].$$

(III) Suppose that $s(\gamma_2)$ is in $\mathcal{B}$ and that there is a marked point A in $\mathcal{B}$ between $s(\gamma_1)$ and $s(\gamma_2)$ such that the arcs f_1 and $s(\gamma_1)A$ are two sides of a triangle, where $f_1, \ldots, f_n$ is a fan as in left hand side of Fig. 14. Then we conclude as in part (a). If no such point A exists then $s(\gamma_1)s(\gamma_2)$ is a boundary segment. We add a marked point A on this boundary arc and argue as in case (II).

(IV) Suppose that $s(\gamma_1) = s(\gamma_2)$. Then there is a triangulation T' containing an arc from some point A on $\mathcal{B}$ (we add the point A if it does not already exist) to B crossing γ_1 and γ_2 and such that locally no other arc in T' crosses both γ_1 and γ_2 and we conclude as above.

Case 2 (ii): $P_1 \neq 0$, $S_1 = 0$ and $P_2 \neq 0$, $S_2 \neq 0$ follows from Case 2(i) by changing the orientation of γ_1 and γ_2.

Case 2 (iii) and (iv): The case $P_2 = 0$ and P_1, S_1, S_2 non-zero and the case $S_2 = 0$ and P_1, S_1, P_2 non-zero follow by similar arguments as above.

Case 3 (i): $P_1 = 0$ and $S_2 = 0$ and S_1 and P_2 non-zero.

(a) Suppose that neither P_2 nor S_1 is a direct string. By a similar argument as in Case 2(i)(a) there are marked points A and B such that there is a triangulation T' of $(S, \mathcal{M})$ containing an arc corresponding to the arc τ_{AB} and we obtain a triangle in $\mathcal{C}(S, \mathcal{M})$

$$\gamma_1 \longrightarrow \gamma_5 \oplus \gamma_6 \longrightarrow \gamma_2 \longrightarrow \gamma_1[1].$$

(b) Suppose that P_2 is a direct string and that S_1 is not a direct string or that P_2 is not a direct string and that S_1 is a direct string. Then we use a similar argument as in case 2(i)(b) above to obtain a triangle in $\mathcal{C}(S, \mathcal{M})$

$$\gamma_1 \longrightarrow \gamma_5 \oplus \gamma_6 \longrightarrow \gamma_2 \longrightarrow \gamma_1[1].$$

(c) Suppose both P_2 and S_1 are direct strings. Then if $s(\gamma_1)s(\gamma_2)$ and $t(\gamma_1)t(\gamma_2)$ are not both boundary segments then the argument is a combination of the above arguments.

Note that if $s(\gamma_1)s(\gamma_2)$ and $t(\gamma_1)t(\gamma_2)$ are boundary segments then we obtain a trivial triangle. Namely, consider a surface $(S, \mathcal{M}')$ where $\mathcal{M}'$ contains 2 more marked points than $\mathcal{M}$, one in each of the boundary segments $s(\gamma_1)s(\gamma_2)$ and $t(\gamma_1)(\gamma_2)$. As before this gives rise to a triangle in $\mathcal{C}(S, \mathcal{M}')$

$$\gamma_1 \longrightarrow \gamma_5 \oplus \gamma_6 \longrightarrow \gamma_2 \longrightarrow \gamma_1[1].$$

Applying the construction in [30] and cutting twice, we obtain the trivial triangle in $\mathcal{C}(S, \mathcal{M})$

$$\gamma_1 \longrightarrow \gamma_1 \longrightarrow 0 \longrightarrow \gamma_2.$$

Case 3 (ii): $P_1 = 0$ and $P_2 = 0$ and S_1 and S_2 non-zero, follows from the above by changing the orientation of γ_1 and γ_2.

Case 3 (iii) and (iv): The case $P_1 = 0$ and $P_2 = 0$ and S_2 and P_2 non-zero, and the case $P_2 = 0$ and $S_2 = 0$ and P_1 and S_1 non-zero, follow by similar arguments to the above.

4.1.2. Arrow crossing

Here we consider the case that M_1 crosses M_2 in an arrow.

Case 1: Suppose the crossing occurs in an inner triangle of T. Let τ be the arc corresponding to the segment $s(\gamma_2)t(\gamma_1) = BC$, see Fig. 15 (a).

Flipping τ to τ' in its quadrilateral gives rise to an overlap given by τ' and we use the module crossing methods above to obtain a triangle in $\mathcal{C}(S, \mathcal{M})$

$$\gamma_1 \longrightarrow \gamma_5 \oplus \gamma_6 \longrightarrow \gamma_2 \longrightarrow \gamma_1[1].$$

We remark that the points A, B and C are not necessarily distinct.

Case 2: Suppose the crossing occurs in a triangle of T where the segment $s(\gamma_2)t(\gamma_1) = BC$ is a boundary segment, see Fig. 15 (b).

Consider the surface $(S, \mathcal{M}')$ where $\mathcal{M}' = \mathcal{M} \cup \{X\}$ and X lies on the boundary segment BC, see Fig. 15 (b). We complete T to a triangulation of $(S, \mathcal{M}')$ by adding an arc τ corresponding to the segment AX. This gives rise to an overlap given by τ. Again we use the module crossing methods above to obtain a triangle in $\mathcal{C}(S, \mathcal{M}')$

$$\gamma_1 \longrightarrow \gamma_5 \oplus \gamma_6 \longrightarrow \gamma_2 \longrightarrow \gamma_1[1].$$

We apply the construction in [30] and cut along τ'. Then γ_6 corresponds to a boundary segment and as above we obtain a triangle in $\mathcal{C}(S, \mathcal{M})$

$$\gamma_1 \longrightarrow \gamma_5 \longrightarrow \gamma_2 \longrightarrow \gamma_1[1].$$

Note that the points B and C may coincide.

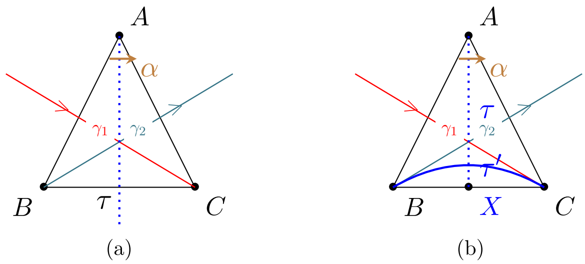


Fig. 15. (a) Arrow crossing in an inner triangle, (b) arrow crossing in a triangle where BC is a boundary segment.

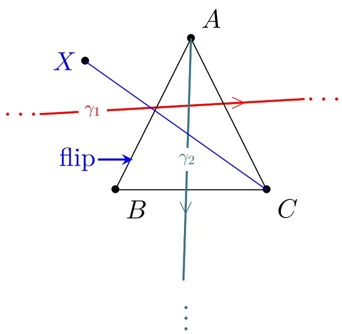


Fig. 16. *Case 1:* 3-cycle crossing.

4.1.3. 3-cycle crossing

Here we consider the case that M_1 crosses M_2 in a 3-cycle.

Case 1: Suppose that $s(\gamma_1) \neq s(\gamma_2)$, see Fig. 16.

Remark that $s(\gamma_1)$ and $s(\gamma_2)$ may or may not lie in the same boundary component of $(S, \mathcal{M})$. In either case we can add a marked point X to obtain a surface $(S, \mathcal{M}')$ such that the segment XC lies between the segments AC and BC. We flip the arc corresponding to the segment AB. This gives rise to a crossing of γ_1 and γ_2 with overlap corresponding to XC. Applying the module crossing methods described above, we obtain two triangles in $\mathcal{C}(S, \mathcal{M}')$. By [30] we obtain the corresponding triangles in $\mathcal{C}(S, \mathcal{M})$.

Case 2: Suppose that $s(\gamma_1) = s(\gamma_2)$.

Denote by $P = \gamma_1(t_0)$ for some $t_0 \in [0, 1]$ the intersection of γ_1 and γ_2 corresponding to the crossing of γ_1 and γ_2 under consideration. Without loss of generality, assume that P is also equal to $\gamma_2(t_0)$. Consider the closed curve $\sigma = \sigma[0, 1]$ which is a union of the segments $\sigma[0, t_0]$ and $\sigma[t_0, 1]$ such that $\sigma[0] = \sigma[1]$ and where $\sigma[0, t_0] = \gamma_1[0, t_0]$ and $\sigma[t_0, 1] = \gamma_2^{-1}[t_0, 0]$. Then σ cannot be homotopic to a point, since otherwise there would not be a 3-cycle crossing. Therefore the local configuration in this case corresponds to one of the two cases illustrated in Fig. 17.

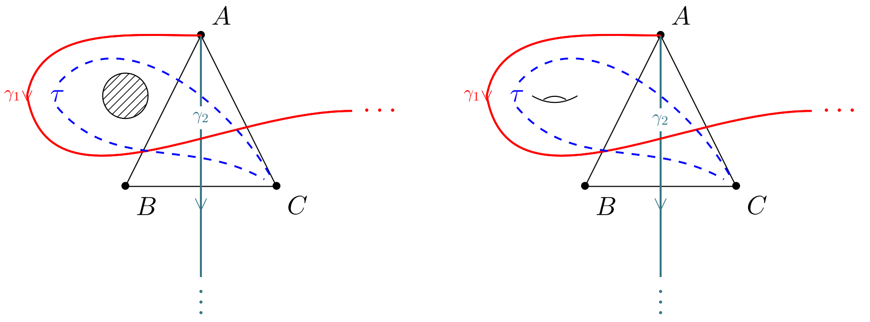


Fig. 17. Possible 3-cycle crossing: $A = s(\gamma_1) = s(\gamma_2)$.

Consider now a curve $\tau = \tau[0,1]$ which is a union of the segments $\tau[0,t_0]$, $\tau[t_0,t_1]$ and $\tau[t_1,1]$ where $\tau[0,t_0] = CP$, $\tau[t_0,t_1]$ is a non-contractible non-selfcrossing curve such that $\tau[t_0] = P = \tau[t_1]$ and $\tau[t_1,1] = CP$. Note that we choose CP to be a curve with no self-intersection. Then $\tau[0,1]$ is homotopic to a closed curve on S without any self-intersection in its interior, see Fig. 17.

There exists a triangulation T' containing the arc corresponding to σ. Thus γ_1 and γ_2 cross in a non-self-crossing overlap containing at least σ. The rest follows as in *Case 1* for module crossings.

The points A, B and C may coincide two by two.

This completes the proof of Theorem 4.1. $\square$

5. Example

Let J be the Jacobian algebra corresponding to the triangulation in Fig. 18. We see that the example contains two crossing arcs γ_1 and γ_2 such that the corresponding string modules M_1 and M_2 cross four times and such that M_1 has one self-crossing. Each type of crossing (in a module, see crossings ①, ② and ⑤, in an arrow, see crossing ④, in a 3-cycle, see crossing ③) occurs at least once. We remark that there is a module crossing in both directions, that is M_1 crosses M_2 in a module, see crossing ①, and M_2 crosses M_1 in a module, see crossing ②.

The arc γ_1 corresponds to the string module $M_1 = M(w_1)$ and γ_2 corresponds to $M_2 = M(w_2)$ where

$$w_1 = \begin{smallmatrix} & & & & 4 & & & & \\ 1 & & & 3 & & 5 & & & 2 \\ & & 2 & & & & & 6 & \end{smallmatrix} \qquad \text{and} \qquad w_2 = \begin{smallmatrix} & & & & 8 & \\ 6 & & & 4 & & 7 \\ & & 3 & & & \end{smallmatrix}.$$

For each of the five crossings ①-⑤, we now explicitly give the modules $M_3 = M(w_3), M_4 = M(w_4), M_5 = M(w_5)$ and $M_6 = M(w_6)$ corresponding to the smoothing of each of the given crossings as defined in Definition 3.5(1). Note that $M_3, \ldots, M_6$ depend on (and therefore change with) the given crossing whereas M_1 and M_2 do not. Similarly, in terms of the corresponding arcs $\gamma_1 = \gamma(w_1), \ldots, \gamma_6 = \gamma(w_6)$, the arcs γ_1 and γ_2 are fixed whereas $\gamma_3, \ldots, \gamma_6$ depend on the crossing under consideration.

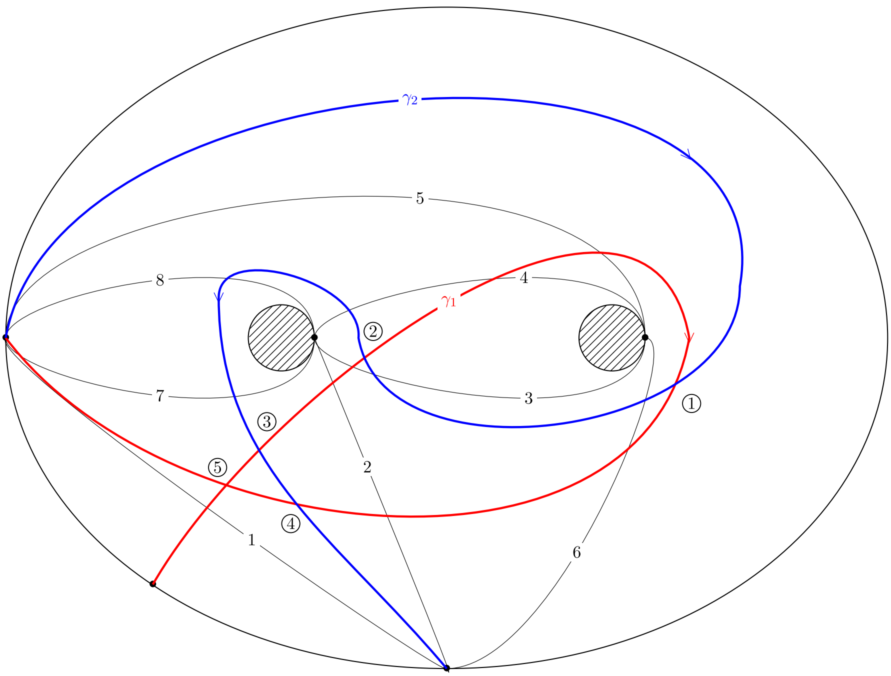


Fig. 18. Triangulation of a surface with two arcs crossing each other 4 times and with one arc with one self-crossing.

Crossing ①: M_1 crosses M_2 in the simple module $\mathbf{6}$ and

$$w_3 = 1\,{}_{2}\,3\,{}^{4}\,5\,\mathbf{6}\,{}_{3}\,4\,{}^{8}\,7 \qquad w_4 = \mathbf{6}\,{}^{2} \qquad w_5 = 1\,{}_{2}\,3\,{}^{4} \qquad w_6 = {}_{2}\,3\,4\,{}^{8}\,7$$

Crossing ②: M_2 crosses M_1 in the module ${}_{\mathbf{3}}\mathbf{4}$ and

$$w_3 = 6\,\mathbf{3}\,\mathbf{4}\,5\,{}_{6}\,2 \qquad w_4 = 1\,{}_{2}\,\mathbf{3}\,\mathbf{4}\,{}^{8}\,7 \qquad w_5 = 6\,2\,{}^{1} \qquad w_6 = {}_{7}\,8\,{}^{5}\,6\,{}^{2}$$

Crossing ③: M_1 crosses $M(w_2^{-1}) \simeq M_2$ in the 3-cycle
$$\begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[dl] \\ 7 \arrow[u] & \end{tikzcd}$$
and

$$w_3 = {}_{1}\,7\,{}^{8}\,4\,{}_{3}\,6 \qquad w_4 = 3\,{}^{4}\,5\,{}_{6}\,2 \qquad w_5 = 0 \qquad w_6 = 6\,{}_{3}\,4\,{}^{8}\,7\,2\,3\,{}^{4}\,5\,{}_{6}\,2$$

Crossing ④: M_1 crosses $M(w_2^{-1}) \simeq M_2$ in the arrow $2 \xrightarrow{\alpha} 7$ and

$$w_3 = 1\,{}_{2}\,3\,{}^{4}\,5\,{}_{6}\,2\,{}_{7}\,{}^{8}\,4\,{}_{3}\,6 \qquad w_4 = 0 \qquad w_5 = 1\,{}_{2}\,3\,{}^{4}\,5 \qquad w_6 = 4\,{}_{3}\,6$$

Crossing ⑤: M_1 crosses itself, that is M_1 crosses $M(w_1^{-1}) \simeq M_1$, in the simple module $\mathbf{2}$ and

$$w_3 = \begin{smallmatrix} 1 & & & & 4 & & & \\ & \mathbf{2} & & 5 & & 3 & & 1 \\ & & 6 & & & & 2 & \end{smallmatrix} \qquad w_4 = \begin{smallmatrix} & & 4 & & & \\ & 3 & & 5 & & 2 \\ \mathbf{2} & & & & 6 & \end{smallmatrix} \qquad w_5 = 0 \qquad w_6 = \begin{smallmatrix} & & & & & & & 4 & & & \\ & & & & & & 5 & & 3 & & 1 \\ & & & 4 & & 6 & & & & 2 & \\ 2 & & 5 & & 3 & & & & & & \\ & 6 & & & & & & & & & \end{smallmatrix}$$

Smoothing crossings ①, ③ and ④ gives rise to two triangles in the cluster category

$$\gamma_2 \longrightarrow \gamma_3 \oplus \gamma_4 \longrightarrow \gamma_1 \longrightarrow \gamma_2[1],$$
$$\gamma_1 \longrightarrow \gamma_5 \oplus \gamma_6 \longrightarrow \gamma_2 \longrightarrow \gamma_1[1],$$

where in the case of crossing ③ the indecomposable object in the cluster category corresponding to γ_5 is the zero object and where in the case of crossing ④ the arc γ_4 corresponds to the arc labelled '1' in the triangulation.

Smoothing crossing ② gives rise to two triangles in the cluster category

$$\gamma_1 \longrightarrow \gamma_3 \oplus \gamma_4 \longrightarrow \gamma_2 \longrightarrow \gamma_1[1],$$
$$\gamma_2 \longrightarrow \gamma_5 \oplus \gamma_6 \longrightarrow \gamma_1 \longrightarrow \gamma_2[1].$$

Smoothing crossing ⑤ gives rise to two triangles in the cluster category

$$\gamma_1 \longrightarrow \gamma_3 \oplus \gamma_4 \longrightarrow \gamma_1 \longrightarrow \gamma_1[1],$$
$$\gamma_1 \longrightarrow \gamma_6 \longrightarrow \gamma_1 \longrightarrow \gamma_1[1].$$

In the Jacobian algebra, smoothing crossings ① and ④ gives rise to the following short exact sequences

$$0 \longrightarrow \begin{smallmatrix} & & & 8 & \\ \mathbf{6} & & 4 & & 7 \\ & 3 & & & \end{smallmatrix} \longrightarrow \begin{smallmatrix} & & & 4 & & & & & & \\ 1 & & 3 & & 5 & & & & 8 & \\ & 2 & & & & \mathbf{6} & & 4 & & 7 \\ & & & & & & 3 & & & \end{smallmatrix} \oplus \begin{smallmatrix} & 2 \\ \mathbf{6} & \end{smallmatrix} \longrightarrow \begin{smallmatrix} & & & 4 & & & \\ 1 & & 3 & & 5 & & 2 \\ & 2 & & & & \mathbf{6} & \end{smallmatrix} \longrightarrow 0,$$

$$0 \longrightarrow \begin{smallmatrix} & 8 & & & \\ 7 & & 4 & & 6 \\ & & & 3 & \end{smallmatrix} \longrightarrow \begin{smallmatrix} & & & 4 & & & & & & & & \\ 1 & & 3 & & 5 & & 2 & & 8 & & & \\ & 2 & & & & 6 & & 7 & & 4 & & 6 \\ & & & & & & & & & & 3 & \end{smallmatrix} \longrightarrow \begin{smallmatrix} & & & 4 & & & \\ 1 & & 3 & & 5 & & 2 \\ & 2 & & & & 6 & \end{smallmatrix} \longrightarrow 0,$$

respectively. However, crossings ② and ③ do not give short exact sequences from M_2 to M_1. Therefore, as stated in Corollary 4.4, $\dim \operatorname{Ext}^1_J(M_1, M_2) = 2$.

Crossing ② is the only crossing that gives a short exact sequence from M_1 to M_2

$$0 \longrightarrow \begin{smallmatrix} & & & \mathbf{4} & & & \\ 1 & & \mathbf{3} & & 5 & & 2 \\ & 2 & & & & 6 & \end{smallmatrix} \longrightarrow \begin{smallmatrix} & & & & 8 & \\ & & & \mathbf{4} & & 7 \\ 1 & & \mathbf{3} & & & \\ & 2 & & & & \end{smallmatrix} \oplus \begin{smallmatrix} 6 & & \mathbf{4} & & & \\ & \mathbf{3} & & 5 & & 2 \\ & & & & 6 & \end{smallmatrix} \longrightarrow \begin{smallmatrix} & & & 8 & \\ 6 & & \mathbf{4} & & 7 \\ & \mathbf{3} & & & \end{smallmatrix} \longrightarrow 0.$$

Therefore $\dim \operatorname{Ext}^1_J(M_2, M_1) = 1$. And finally, we have

$$\dim \operatorname{Ext}^1_J(M_1, M_2) + \dim \operatorname{Ext}^1_J(M_2, M_1) = 3 = \operatorname{Int}(\gamma_1, \gamma_2) - k$$

where $\operatorname{Int}(\gamma_1, \gamma_2) = 4$ and $k = 1$ corresponds to the only 3-cycle crossing corresponding to crossing ③.

Crossing ⑤ gives a short exact sequence from M_1 to M_1

$$0 \longrightarrow \begin{smallmatrix} &&&4&&&\\ \mathbf{2}&&5&&3&&1\\ &6&&&&2& \end{smallmatrix} \longrightarrow \begin{smallmatrix} 1&&&&4&&&\\ &\mathbf{2}&&5&&3&&1\\ &&6&&&&2& \end{smallmatrix} \oplus \begin{smallmatrix} &&4&&&\\ &3&&5&&2\\ \mathbf{2}&&&&6& \end{smallmatrix} \longrightarrow \begin{smallmatrix} &&&4&&&\\ 1&&3&&5&&2\\ &\mathbf{2}&&&&6& \end{smallmatrix} \longrightarrow 0.$$

Since crossing ⑤ is the only self-crossing of γ_1, we have $\dim \operatorname{Ext}^1_J(M_1, M_1) = 1$.

Since γ_2 has no self-crossings, we have $\dim \operatorname{Ext}^1_J(M_2, M_2) = 0$.

Appendix A. On the canonicity of the generalised cluster category associated with a surface

Claire Amiot

Let k be an algebraically closed field and $(S, \mathcal{M})$ be a marked surface (such that all marked points are in the boundary of S). The cluster category $\mathcal{C}(S, \mathcal{M})$ associated to the marked surface $(S, \mathcal{M})$ is defined to be the generalised cluster category $\mathcal{C}_T := \mathcal{C}(Q(T), W(T))$ (as defined in [1]) where T is a triangulation of $(S, \mathcal{M})$ and $(Q(T), W(T))$ is the quiver with potential associated to T by [26]. This category is well defined in the following sense: if T' is another triangulation of $(S, \mathcal{M})$, then combining the main results of [26] and [23] one gets an equivalence of triangulated categories $\mathcal{C}_T \simeq \mathcal{C}_{T'}$. This implies that $\mathcal{C}(S, \mathcal{M})$ is only well-defined up to equivalence of categories. Indeed, a priori given T and T' there is no canonical equivalence $\mathcal{C}_T \simeq \mathcal{C}_{T'}$. The aim of this appendix is to exhibit some questions and problems this non-canonicity may pose.

More precisely let us recall the following result due to Brüstle and Zhang.

Theorem A.1. *[6, Thm 1.1] Let $(S, \mathcal{M})$ be a marked surface such that all marked points are on the boundary of S. A parametrization of the isoclasses of indecomposable objects in $\mathcal{C}(S, \mathcal{M})$ is given by string objects and band objects, where*

(1) *the string objects are indexed by the homotopy classes of non-contractible curves in $(S, \mathcal{M})$ with end points in $\mathcal{M}$ which are not homotopic to a boundary segment of $(S, \mathcal{M})$, subject to the equivalence relation $\gamma \sim \gamma^{-1}$;*
(2) *the band objects are indexed by $k^* \times \pi_1^*(S)/\sim$ where $\pi_1^*(S)/\sim$ is given by the nonzero elements of the fundamental group of S subject to the equivalence relation generated by cyclic permutation and $\gamma \sim \gamma^{-1}$.*

Regarding this result one could first think that we get a description of the objects of $\mathcal{C}(S, \mathcal{M})$ independent of the choice of a triangulation. However the parametrization depends on the choice of a triangulation. So given a triangulation T let us denote by s^T (resp. b^T) the above bijections that send a curve (resp. a curve with a scalar) to an indecomposable object in $\mathcal{C}_T$. The different facts shown in this appendix can be summarized as the following:

Proposition A.2. *Let $(S, \mathcal{M})$ be a marked surface such that all marked points are on the boundary of S.*

(1) *There exist triangulations T and T', and an equivalence $\Phi : \mathcal{C}_T \to \mathcal{C}_{T'}$ of triangulated categories that sends a string object to a band object.*
(2) *For any triangulations T and T', there exists an equivalence $\Phi : \mathcal{C}_T \to \mathcal{C}_{T'}$ of triangulated categories such that $\Phi \circ s^T = s^{T'}$.*
(3) *For the equivalences Φ of (2), we may have $\Phi \circ b^T \neq b^{T'}$.*

As a consequence of (2) we obtain a bijection between string objects in $\mathcal{C}(S, \mathcal{M})$ and homotopy classes of curves in $(S, \mathcal{M})$ that does not depend on the choice of a triangulation.

Remark A.3. A generalisation of the bijection s^T has been constructed in [38] in the case where $(S, \mathcal{M})$ is a marked surface with punctures and T is an admissible triangulation (see [38, Def 5]). In the same paper, the authors state a more general analogue of (2) in the case where T and T' are admissible [38, Thm 3.10]. The proof given there is not fully detailed. First, the fact that there exists a canonical equivalence between $\mathcal{C}_T$ and $\mathcal{C}_{T'}$ [38, (3.3) in subsection 3.1] is used without being proved. Indeed the equivalence constructed in [23] is not canonical since it depends of a choice of a right equivalence (see next subsection for more details). Secondly the proof in [38, Appendix C] does not stress the importance of signs in the computation of the mutation of decorated representations in the sense of [12]. Point (3) of Proposition A.2 above and subsection A.3.2 below show that the manipulation of signs is actually a subtle issue in the computation. Though the results presented here are not strictly speaking original, I have thought it would be useful to the community to clarify the aforementioned issues.

A.1. The problem on reduction

A.1.1. Construction of the equivalence $\mathcal{C}_T \to \mathcal{C}_{T'}$

Let T and T' be triangulations of $(S, \mathcal{M})$, and $\mathbf{s}$ be a sequence of flips such that $T' = \mathfrak{f}_{\mathbf{s}}(T)$. Then Labardini constructed in [26] a right equivalence between the associated quivers with potentials:

$$\varphi_{\mathbf{s}} : (Q(T'), W(T')) \to \mu_{\mathbf{s}}(Q(T), W(T)),$$

where $\mu_{\mathbf{s}}$ is the mutation of the quiver with potential defined by Derksen, Weyman and Zelevinsky in [12]. Then by [23], there exists an equivalence $\Phi_{\mathbf{s}} : \mathcal{C}_T \to \mathcal{C}_{T'}$. This equivalence depends not only on the choice of the sequence of flips $\mathbf{s}$ but it also depends on the choice of a right equivalence at each flip.

Let us concentrate on this second dependence and assume that $\mathbf{s} = \mathbf{i}$, that is T and T' differ only by one flip of an arc $\mathbf{i}$. Denote by $(Q, W) := (Q(T), W(T))$ the QP associated with T and by $(\widetilde{Q'}, \widetilde{W'}) := \widetilde{\mu}_{\mathbf{i}}(Q, W))$ the unreduced mutation of (Q, W) at vertex $\mathbf{i}$. Now fix the right equivalence $\varphi : (\widetilde{Q'}, \widetilde{W'}) \to (Q', W') = (Q(T'), W(T'))$ corresponding to a reduction of the quiver with potential $(\widetilde{Q'}, \widetilde{W'})$. The functor $\Phi_{\mathbf{i}}$ constructed by Keller and Yang is the composition of two equivalences:

$$\mathcal{C}_T \xrightarrow{\widetilde{\Phi_{\mathbf{i}}}} \mathcal{C}_{(\widetilde{Q'}, \widetilde{W'})} \xrightarrow{R^\varphi} \mathcal{C}_{T'} \ ,$$

where R^φ is induced by φ_* on the corresponding Ginzburg DG algebras (see [23, Lemma 2.9]).

A.1.2. Link with mutations of (decorated) representations

The authors in [12] define a notion of mutation of decorated representations of a (nondegenerate) quiver with potential: for a module $M \in \operatorname{mod} \ \mathrm{J}(Q, W)$ and a vertex $\mathbf{i}$ of Q, they define a module $\widetilde{\mu}_{\mathbf{i}}(M) \in \operatorname{mod} \ \mathrm{J}(\widetilde{Q'}, \widetilde{W'})$. Then any reduction $\varphi : (\widetilde{Q'}, \widetilde{W'}) \to (Q', W')$ induces an isomorphism of algebras $\varphi : \mathrm{J}(\widetilde{Q'}, \widetilde{W'}) \to \mathrm{J}(Q', W')$ by [12, Prop. 3.7], and so an equivalence

$$\operatorname{mod} \ \mathrm{J}(\widetilde{Q'}, \widetilde{W'}) \xrightarrow{\varphi_*} \operatorname{mod} \ \mathrm{J}(Q', W') \ .$$

Note that in [12, Def. 10.4] the equivalence is defined to be the restriction functor $(\varphi^{-1})^*$ but since φ is an isomorphism of algebras we have $(\varphi^{-1})^* = \varphi_*$.

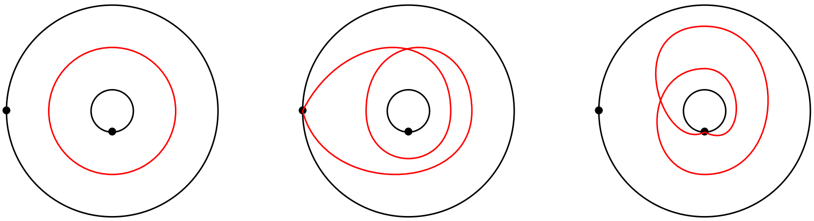
The mutation $\mu_{\mathbf{i}}(M)$ of M at $\mathbf{i}$ is then defined to be $\varphi_*(\widetilde{\mu}_{\mathbf{i}}(M))$. This implies that $\mu_{\mathbf{i}}(M)$ is only defined up to right equivalence of representation (cf. Remark 10.3 in [12]) and not up to isomorphism of module.

By construction the cluster category $\mathcal{C}_T$ comes with a canonical cluster-tilting object that we denote by X, with the property $\operatorname{End}_{\mathcal{C}_T}(X) \simeq \mathrm{J}(Q, W)$. The functor $F := \operatorname{Hom}_{\mathcal{C}_T}(X[-1], -) : \mathcal{C}_T \to \operatorname{mod} \ \mathrm{J}(Q, W)$ is dense and sends any indecomposable object not isomorphic to a summand of X to an indecomposable module. Similarly we denote by $\widetilde{X'}$ (resp. X') the canonical cluster-tilting objects in $\mathcal{C}_{(\widetilde{Q'}, \widetilde{W'})}$ (resp. in $\mathcal{C}_{T'}$). Then we have the following:

$$\begin{tikzcd}
\mathcal{C}_T \arrow[r, "\widetilde{\Phi_{\mathbf{i}}}"] \arrow[d, "F"'] & \mathcal{C}_{(\widetilde{Q'}, \widetilde{W'})} \arrow[r, "R^\varphi"] \arrow[d, "\widetilde{F'}"] & \mathcal{C}_{T'} \arrow[d, "F'"] \\
\operatorname{mod} \ \mathrm{J}(Q, W) \arrow[r, dashed, "\widetilde{\mu}_{\mathbf{i}}"] & \operatorname{mod} \ \mathrm{J}(\widetilde{Q'}, \widetilde{W'}) \arrow[r, "\varphi_*"] & \operatorname{mod} \ \mathrm{J}(Q', W')
\end{tikzcd} \quad . \qquad (5)$$

In this diagram, the right hand square is clearly commutative, since $R^\varphi(\widetilde{X'})$ is isomorphic to X' in the category $\mathcal{C}_{T'}$. Moreover Plamondon showed in [36, Prop 4.1] that if M is an object of $\mathcal{C}_T$, then $\widetilde{\mu}_{\mathbf{i}}(F(M))$ and $\widetilde{F'}(\widetilde{\Phi_{\mathbf{i}}}(M))$ are isomorphic in mod $\mathrm{J}(\widetilde{Q'}, \widetilde{W'})$. Hence the left square also commutes.

Since for any reduction φ, if we compose φ with an automorphism of the algebra $\mathrm{J}(Q', W')$ that fixes each vertex we obtain another reduction, it is easy to construct an equivalence $\Phi_{\mathbf{i}} : \mathcal{C}_T \to \mathcal{C}_{T'}$ sending certain string objects to band objects (see Remark 10.3 in [12]). For instance if $(S, \mathcal{M})$ is the annulus with two marked points then for any triangulation T, (Q_T, W_T) is the Kronecker quiver with the zero potential. Denote by a and b its arrows, the automorphism sending a to a and b to $a + b$ sends the string a to a band of type ab^{-1}. The automorphism exchanging a and b sends the string a to the string b.



band ab^{-1} string a string b

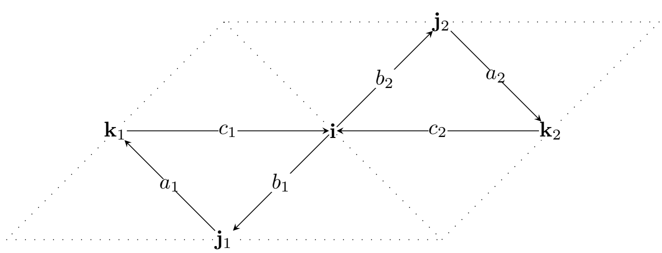
A.2. Canonical bijection for strings

In this subsection, we prove that the bijection s^T constructed in [6] is independent of T as soon as we only allow certain reductions.

A.2.1. Canonical reduction

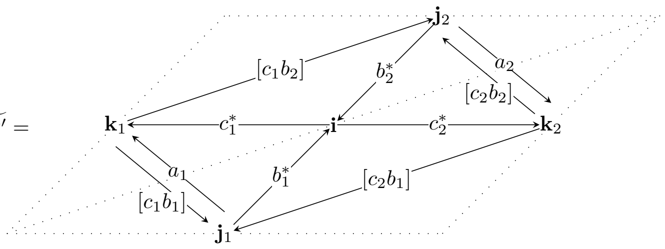
One way to handle this problem is to allow only certain kind of reductions when constructing the triangle equivalences $\Phi_{\mathbf{i}}$. This can be done easily in the case where $(S, \mathcal{M})$ is a marked surface with all marked points are on the boundary. Indeed in this case the quiver with potential is especially simple.

Let T be a triangulation, $\mathbf{i}$ be an arc of T and $T' := \mathfrak{f}_{\mathbf{i}}(T)$ be the flip of T at $\mathbf{i}$. Denote by Δ_1 and Δ_2 the triangles in T having $\mathbf{i}$ as a side. Then Δ_1 and Δ_2 are distinct. If Δ_1 and Δ_2 are internal triangles then locally the quiver with potential (Q_T, W_T) associated to T looks as follows:



with $W = c_1b_1a_1 + c_2b_2a_2$. Note that we may have $\mathbf{j}_1 = \mathbf{j}_2$ (or $\mathbf{k}_1 = \mathbf{k}_2$), but in that case, there is a way to distinguish the arrow b_1 from the arrow b_2 since each arrow is canonically associated to a triangle of the triangulation. By [12] we obtain the following quiver with potential $\widetilde{\mu_i}(Q, W) = (\widetilde{Q'}, \widetilde{W'})$ after 'unreduced' mutation at $\mathbf{i}$.

$\widetilde{Q'} =$



$$\widetilde{W'} = [c_1b_1]a_1 + [c_2b_2]a_2 + [c_1b_1]b_1^*c_1^* + [c_2b_2]b_2^*c_2^* + [c_2b_1]b_1^*c_2^* + [c_1b_2]b_2^*c_1^*.$$

Let $\varphi_{\mathbf{i}}^T : k\widetilde{Q'} \to k\widetilde{Q'}$ be the map defined by

$$\begin{cases} \varphi_{\mathbf{i}}^T(\alpha) = \alpha \text{ if } \alpha \neq a_1, a_2 \\ \varphi_{\mathbf{i}}^T(a_\ell) = a_\ell - b_\ell^* c_\ell^* \text{ for } \ell = 1, 2. \end{cases}$$

It is immediate to see that $\varphi_{\mathbf{i}}^T$ is a right equivalence between the quiver with potential $(\widetilde{Q'}, \widetilde{W'})$ and the direct sum of the quiver with potential (Q', W') with a trivial quiver with potential, where (Q', W') is the quiver with potential associated with the triangulation $T' = \mathfrak{f}_{\mathbf{i}}(T)$. Note that if Δ_1 or Δ_2 have boundary sides, then the arcs $\mathbf{j}_1$, $\mathbf{j}_2$, $\mathbf{k}_1$ and $\mathbf{k}_2$ may not exist, so the quiver with potential (Q_T, W_T) is simpler and the definition of the reduction $\varphi_{\mathbf{i}}^T$ is similar. Moreover in the case where $(\widetilde{Q'}, \widetilde{W'})$ does not have any 2-cycle, then $\varphi_{\mathbf{i}}^T$ is the identity map. This leads to introduce the following.

Definition A.4. For any arc $\mathbf{i}$ in T, the map $\varphi_{\mathbf{i}}^T$ is called the *canonical reduction* at $\mathbf{i}$.

A.2.2. The bijection s^T

Before proving the main result, let us recall the construction of the map s^T, and some properties on string modules that will be used in the proof.

By [2] the Jacobian algebra $\mathrm{J}(Q, W)$ is a string algebra. In such an algebra, a word $w = \alpha_1 \dots \alpha_n$ of arrows or formal inverse of arrows of Q^{op} is called a *string* if $\alpha_{i+1} \neq \alpha_i^{-1}$ and no subword nor its inverse belongs to the Jacobian ideal.

Let $w = \alpha_n \dots \alpha_1$ be a string and $\lambda_1, \dots, \lambda_n$ be in k^*. Define a module $M^T(w : \lambda_1, \dots, \lambda_n)$ in mod $\mathrm{J}(Q, W) = \mathrm{Rep}(Q^{\mathrm{op}}, \partial W)$ as follows: For any $\ell = 0 \dots, n$ let M_ℓ be a 1-dimensional k-vector space. Then for any vertex $\mathbf{i}$ in Q_0 set

$$M_{\mathbf{i}}^T = \begin{cases} \bigoplus_{\ell, t(\alpha_\ell) = \mathbf{i}} M_\ell \oplus M_0 & \text{if } s(\alpha_1) = \mathbf{i} \\ \bigoplus_{\ell, t(\alpha_\ell) = \mathbf{i}} M_\ell & \text{else} \end{cases}$$

For any arrow $\alpha : \mathbf{i} \to \mathbf{j}$ in Q_1^{op}, if there exists ℓ such that $\alpha = \alpha_\ell$ (resp. $\alpha^{-1} = \alpha_\ell$) then $M_{\ell-1}$ is a direct summand of $M_{\mathbf{i}}^T$ (resp. $M_{\mathbf{j}}^T$) and M_ℓ a summand of $M_{\mathbf{j}}^T$ (resp. $M_{\mathbf{i}}^T$) and the restriction of M_α^T to $M_{\ell-1}$ (resp. M_ℓ) is the multiplication by λ_ℓ from $M_{\ell-1}$ to M_ℓ (resp. the multiplication by λ_ℓ^{-1} from M_ℓ to $M_{\ell-1}$).

Definition A.5. The *string module* associated to w is defined to be $M^T(w) := M^T(w : 1, \ldots, 1)$.

The following isomorphisms are easy to check and classical [5].

Lemma A.6. *We have the following isomorphisms in* mod $\mathrm{J}(Q, W)$*:*

$$\begin{aligned} M^T(w : \lambda_1, \ldots, \lambda_n) &= M^T(w^{-1} : \lambda_n^{-1}, \ldots, \lambda_1^{-1}) \\ &\simeq M^T(w : 1, \lambda_2, \ldots, \lambda_n) \\ &\simeq M^T(w : \lambda_1, \ldots, \lambda_{\ell-1}\lambda_\ell, 1, \lambda_{\ell+1}, \ldots, \lambda_n) \ \forall \ell \end{aligned}$$

Corollary A.7. *The module $M^T(w : \lambda_1, \ldots, \lambda_n)$ is isomorphic to the string module $M^T(w) = M^T(w^{-1})$.*

Now let γ be an oriented curve on S with endpoints in $\mathcal{M}$ which is not homotopic to a boundary segment or to an arc of T. Up to homotopy, we may assume that γ intersects each arc of T transversally and does not cut an arc of T twice in succession. Then one can associate to γ a sequence w of arrows or inverse arrows in Q_1^{op} corresponding to the angles of T intersected by γ. The map $\gamma \to w(\gamma)$ is shown to be a bijection between nontrivial homotopy classes of such oriented curves in $(S, \mathcal{M})$ and strings in $\mathrm{J}(Q, W)$ in [2]. Moreover, the string associated to γ^{-1} is clearly $w(\gamma)^{-1}$.

The bijection s^T is defined in [6] as follows: if $\gamma = \mathbf{i}$ is an arc of T, then $s^T(\gamma)$ is defined to be the object $X_{\mathbf{i}}$ which is the indecomposable summand of the canonical cluster-tilting object X in $\mathcal{C}_T$ corresponding to the vertex $\mathbf{i}$ of Q; if γ is not an arc of T, then $s^T(\gamma)$ is the indecomposable object $X(\gamma)$ such that $F(X(\gamma)) \simeq M^T(w(\gamma))$ in mod $\mathrm{J}(Q, W)$. This indecomposable object is unique up to isomorphism.

A.2.3. Compatibility for strings

The following result is the main result of this appendix.

Theorem A.8. *Let T and T' be triangulations of a marked surface with marked points on the boundary, and s^T and $s^{T'}$ be the bijections described above. For a sequence of flips $\mathbf{s}$ such that $\mathfrak{f}_{\mathbf{s}}(T) = T'$, denote by $\Phi_{\mathbf{s}} : \mathcal{C}_T \to \mathcal{C}_{T'}$ the equivalence defined in [23] where at each mutation we apply the canonical reduction. Then for any such sequence $\mathbf{s}$ we have*

$$\Phi_{\mathbf{s}} \circ s^T = s^{T'}.$$

Proof. Let T be a triangulation of $(S, \mathcal{M})$ and $\mathbf{i}$ be an arc of T. Denote by T' the triangulation $\mathfrak{f}_{\mathbf{i}}(T)$. Denote by φ the canonical reduction $(\widetilde{Q'}, \widetilde{W'}) \to (Q', W')$ defined above. It is enough to show that $R^\varphi \circ \widetilde{\Phi_{\mathbf{i}}} \circ s^T = s^{T'}$.

Let γ be a curve on S with endpoints in $\mathcal{M}$. We consider the following cases.

Case 1: $\gamma = \mathbf{j}$ is an arc of T and T'. Then we have $s^T(\mathbf{j}) = X_{\mathbf{j}}$ and from the definition of $\widetilde{\Phi_{\mathbf{i}}}$ we have $\widetilde{\Phi_{\mathbf{i}}}(X_{\mathbf{j}}) \simeq \widetilde{X'}_{\mathbf{j}}$ and $R^\psi \circ \widetilde{\Phi_{\mathbf{i}}}(X_{\mathbf{j}}) \simeq X'_{\mathbf{j}} = s^{T'}(\mathbf{j})$ for any choice of reduction ψ.

Case 2: $\gamma = \mathbf{i}$. Then $s^T(\mathbf{i}) = X_{\mathbf{i}}$. Denote by $\mathbf{i'}$ the arc of T' which is not in T. By definition, for any choice of reduction ψ, the object $R^\psi \circ \widetilde{\Phi_{\mathbf{i}}}(X_{\mathbf{i}})$ is the cone of the map

$$X'_{\mathbf{i'}} \to \bigoplus_{\mathbf{i'} \to k \in Q'_1} X'_k$$

which is isomorphic to the cocone of the map

$$\bigoplus_{j \to \mathbf{i'} \in Q'_1} X'_j \to X'_{\mathbf{i'}}$$

by the properties of exchange triangles in a 2-Calabi–Yau category with cluster-tilting objects. Applying the functor $F' = \mathrm{Hom}_{\mathcal{C}_{T'}}(X'[-1], -)$ we obtain that $F'(\Phi_{\mathbf{i}}(X_{\mathbf{i}}))$ is isomorphic to the cokernel of the map

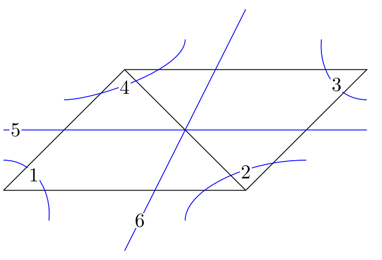
$$\bigoplus_{j \to \mathbf{i'} \in Q'_1} P'_j \to P'_{\mathbf{i'}}$$

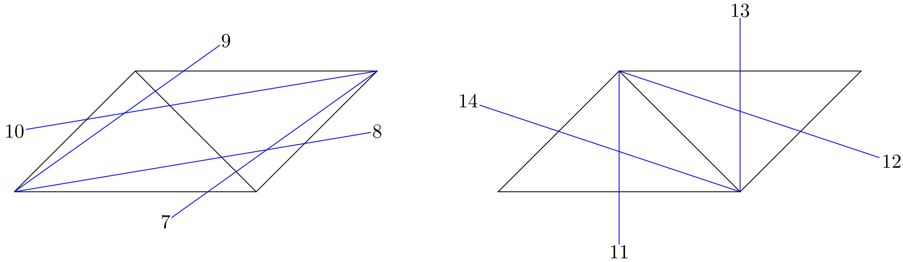
where P'_ℓ is the projective associated to the vertex ℓ in $\mathrm{J}(Q', W')$. Therefore $F'(\Phi_{\mathbf{i}}(X_{\mathbf{i}}))$ is isomorphic to the simple module $S_{\mathbf{i'}}$ of $\mathrm{J}(Q', W')$ associated to the vertex $\mathbf{i'}$, which is the module $M^{T'}(\mathbf{i})$.

Case 3: $\gamma = \mathbf{i'}$. This case is similar to the previous case.

Case 4: γ is not an arc of T and not an arc of T'. Denote by w (resp. w') the string in $\mathrm{J}(Q, W)$ (resp. $\mathrm{J}(Q', W)$) corresponding to γ. Then by the commutative diagram (5) and the definition of the bijections s^T and $s^{T'}$, it is enough to show that $\varphi_*(\widetilde{\mu_{\mathbf{i}}} M^T(w))$ is isomorphic to $M^{T'}(w')$ in mod $\mathrm{J}(Q', W')$.

As in the previous section, we denote by Δ_1 and Δ_2 the triangles of T sharing $\mathbf{i}$. There is exactly 14 ways for γ to cross $\Delta_1 \cup \Delta_2$, which are described in the following pictures:

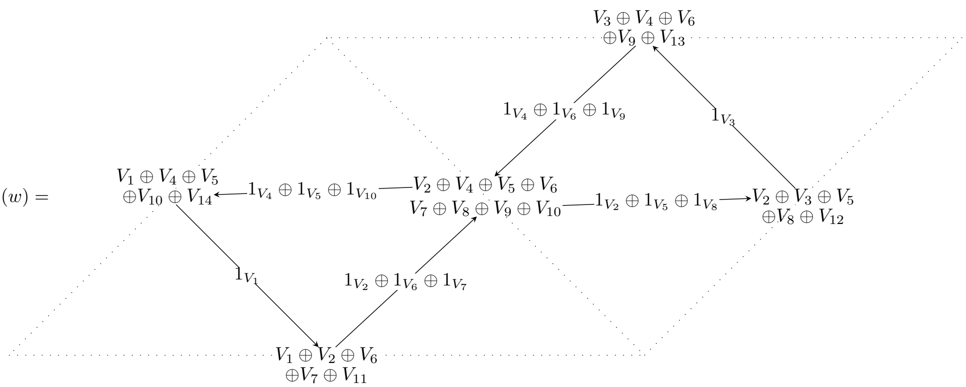




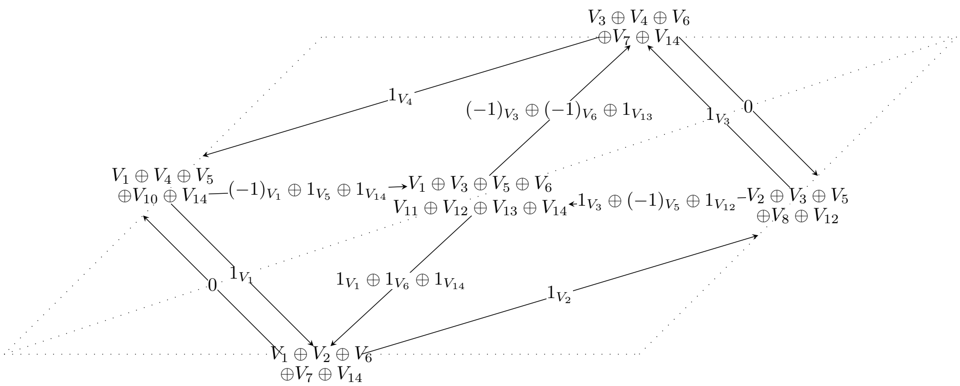
Denote by n_ℓ, $\ell = 1, \dots, 14$ the number of times γ intersects $\Delta_1 \cup \Delta_2$ in the way ℓ (in both directions), and define $V_\ell = k^{n_\ell}$. Note that $\sum_{\ell=7}^{14} n_\ell \leq 2$ since these crossings correspond to endpoints of γ.

Then the restriction of the representation $M^T(w)$ to the quiver $Q_{\Delta_1 \cup \Delta_2}$ is the following:

$M^T(w) =$



Then a direct calculation gives the following representation for $\widetilde{\mu_{\mathbf{i}}}(M^T(w))$:



Note that in the computation of $\widetilde{\mu}_{\mathbf{i}}(M^T(w))$, the splitting data (10.8) and (10.9) of [12] are always 0 or identity, hence the representation does not depend on the choice of these data.

Applying the canonical reduction we obtain the same representation except that the action of the arrows a_1 and a_2 are 0 instead of 1_{V_3} and 1_{V_2}. This representation is of the form $M^{T'}(w' : \lambda_1, \ldots, \lambda_n)$ with $\lambda_\ell = \pm 1$ so using Corollary A.7 we obtain the isomorphism

$$\varphi_*(\widetilde{\mu}_{\mathbf{i}}(M^T(w)) \simeq M^{T'}(w') \quad \text{in mod } \mathrm{J}(Q', W')$$

which ends Case 4 and the proof. □

Remark A.9. The same kind of questions can be asked in the case where $(S, \mathcal{M})$ is a surface with punctures. As mentioned above in Remark A.3, if T and T' are admissible triangulations (that are triangulations where every puncture is in a self-folded triangle), then a similar result has been stated in [38]. But an analogue of the canonical reduction needs to be defined in the case where T and T' are linked by a $\diamond$-flip.

More generally, if T is any triangulation, $T' = \mathfrak{f}_{\mathbf{i}}(T)$ and γ is an arc (thus without selfcrossings) which is not in T and T', then Labardini defined in [27] an indecomposable module $M^T(\gamma) \in \text{mod } \mathrm{J}(Q, W)$ and shows that $\mu_{\mathbf{i}}(M^T(\gamma))$ is right equivalent to $M^{T'}(\gamma)$. A priori, this does not entirely prove that there is a bijection compatible with any triangulation between arcs and direct summands of cluster-tilting objects in $\mathcal{C}(S, \mathcal{M})$ since $\mu_{\mathbf{i}}(M^T(\gamma))$ is only defined up to right equivalence and not up to isomorphism. In this case, the right equivalence $\mu_{\mathbf{i}}(Q, W) \to (Q', W')$ constructed by Labardini in [26] is much more complicated to describe and so it is not so clear that an analogue of the 'canonical' reduction described in the present work does exist.

A.3. The problem with bands

The aim of this subsection is to show that the situation is not as nice for bands. Before exhibiting counter-examples, let us redefine the bijection b^T in a more precise way than in [2] and [6].

A.3.1. Band modules and the bijection b^T

In a string algebra, a string $b = \alpha_1 \ldots \alpha_n$ is called a *band* if $s(\alpha_1) = t(\alpha_n)$, if any power b^m of b is a string and if b is not a power of any string.

Let $b = \alpha_1 \ldots \alpha_n$ be a band in $\mathrm{J}(Q, W)$ and $\lambda_1, \ldots, \lambda_n$ be in k^*. We define a module $B^T(b : \lambda_1, \ldots, \lambda_n)$ in mod $\mathrm{J}(Q, W)$: For any $\ell \in \mathbb{Z}/n\mathbb{Z}$ let B_ℓ be a 1-dimensional k-vector space. Then for any vertex $\mathbf{i}$ in Q_0 set

$$(B^T)_{\mathbf{i}} := \bigoplus_{\ell, t(\alpha_\ell) = \mathbf{i}} B_\ell.$$

For any arrow $\alpha : \mathbf{i} \to \mathbf{j}$ in Q_1^{op}, if there exists ℓ such that $\alpha = \alpha_\ell$ (resp. $\alpha^{-1} = \alpha_\ell$) then $B_{\ell-1}$ is a direct summand of $(B^T)_{\mathbf{i}}$ (resp. $(B^T)_{\mathbf{j}}$) and B_ℓ a summand of $(B^T)_{\mathbf{j}}$ (resp. $(B^T)_{\mathbf{i}}$); the restriction of $(B^T)_\alpha$ to $B_{\ell-1}$ (resp. B_ℓ) is defined to be the multiplication by λ_ℓ from $B_{\ell-1}$ to B_ℓ (resp. the multiplication by λ_ℓ^{-1} from B_ℓ to $B_{\ell-1}$).

Definition A.10. Let b be a band and $\lambda \in k^*$. The (regular simple) *band module* associated with $(b; \lambda)$ is defined to be $B^T(b; \lambda) := B^T(b : \lambda, 1, \dots, 1)$.

The following is classical and easy to check [5].

Lemma A.11. *Let $b = \alpha_1 \dots \alpha_n$ be a band and denote by $b' := \alpha_2 \dots \alpha_n \alpha_1$. Then for any $\lambda_1, \dots, \lambda_n$ we have isomorphisms*

$$\begin{aligned} B^T(b : \lambda_1, \dots, \lambda_n) &\simeq B^T(b' : \lambda_2, \dots, \lambda_n, \lambda_1) \\ &\simeq B^T(b^{-1} : \lambda_n^{-1}, \dots, \lambda_1^{-1}) \\ &\simeq B^T(b : 1, \lambda_1\lambda_2, \dots, \lambda_n). \end{aligned}$$

Corollary A.12. *The module $B^T(b : \lambda_1, \dots, \lambda_n)$ is isomorphic to $B^T(b; \prod_\ell \lambda_\ell)$ and we have*

$$B^T(b; \lambda) \simeq B^T(b^{-1}; \lambda^{-1}) \simeq B^T(b'; \lambda)$$

for any b' cyclic permutation of b.

Denote by $\pi_1^{\mathrm{free},*}(S)$ the set of nontrivial conjugacy classes of the fundamental group $\pi_1(S)$. This set coincides with the set of non-contractible oriented closed curves on S up to free homotopy. Consider the subset $\pi_1^{\mathrm{free,irred},*}(S) \subset \pi_1^{\mathrm{free}}(S)$ of non-contractible irreducible closed curves γ, that are conjugacy classes of closed curves which are not conjugate to a power of a closed curve. A natural bijection b^T between $\pi_1^{\mathrm{free,irred},*}(S)$ and the set of bands in (Q, W) up to cyclic permutation is described in [2], which sends a curve (transversal to T) to the sequence of arrows (and inverse arrows) of Q_1^{op} corresponding to the sequence of angles of T intersected by γ.

Combining this bijection with the definition of B^T and the construction of all the band modules in [5] (associated with power of bands) we obtain a natural bijection

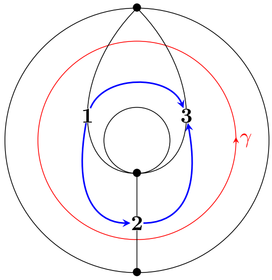
$$b^T : (\pi_1^{\mathrm{free}}(S) \times k^*)/\sim \longrightarrow \{\text{band modules in } \mathrm{J}(Q, W)\}/\mathrm{iso}\ ,$$

where the equivalence relation in $\pi_1^{\mathrm{free}}(S) \times k^*$ is generated by $(\gamma, \lambda) \sim (\gamma^{-1}, \lambda^{-1})$.

Note that this bijection is not exactly the one described in [2,6]. The one in [2,6] needs to make a choice of an orientation for each element in the set $\pi_1^{\mathrm{free,irred}}(S)/\gamma \sim \gamma^{-1}$, choice which is not canonical.

A.3.2. Example

Let $(S, \mathcal{M})$ be the annulus with 3 marked points, and consider the following triangulation T on $(S, \mathcal{M})$ corresponding to the following quiver Q_T^{op}.



Let γ be the following simple generator of $\pi_1(S)$, and λ be in k^*. The module $B^T(\gamma, \lambda)$ is isomorphic to the following representation:

$$B^T(\gamma,\lambda) \simeq \begin{tikzcd} & k \arrow[dr, "\lambda"] & \\ k \arrow[ur, "1"] \arrow[rr, "1"'] & & k \end{tikzcd}$$

Define the triangulations $T^{\mathbf{1}} := \mathfrak{f}_{\mathbf{1}}(T)$ and $T^{\mathbf{2}} := \mathfrak{f}_{\mathbf{2}}(T)$. Then a direct computation gives the isomorphisms:

$$\mu_{\mathbf{1}}(B^T(\gamma,\lambda)) \simeq \begin{tikzcd} & k \arrow[dl, "-1"'] \arrow[dr, "\lambda"] & \\ k & & k \arrow[ll, "1"] \end{tikzcd} \simeq B^{T^{\mathbf{1}}}(\gamma, -\lambda).$$

Note that here, since $\mathbf{1}$ is a sink in Q^{op}, $\widetilde{\mu}_{\mathbf{1}}(Q^{\mathrm{op}}, W)$ is already reduced and so the canonical reduction is the identity morphism.

Other direct computations give the following isomorphisms:

$$\mu_{\mathbf{2}}(B^T(\gamma,\lambda)) \simeq \begin{tikzcd} & 0 \arrow[dl] & \\ k \arrow[rr, shift left, "1"] \arrow[rr, shift right, "\lambda"'] & & k \arrow[ul] \end{tikzcd} \simeq B^{T^{\mathbf{2}}}(\gamma, \lambda).$$

$$\mu_{\mathbf{2}}(B^{T^{\mathbf{2}}}(\gamma,\lambda)) \simeq \begin{tikzcd} & k \arrow[dr, "1"] & \\ k \arrow[ur, "-\lambda"] \arrow[rr, "1"'] & & k \end{tikzcd} \simeq B^{T}(\gamma,-\lambda)$$

This implies that $\mu_{\mathbf{2}}$ (with the canonical reduction) is not an involution, and therefore the autoequivalence $\Phi_{\mathbf{2}}^2$ (defined with the canonical reduction) of $\mathcal{C}_T$ is not isomorphic to the identity functor. This was already noticed in [12, Thm 10.13, (10.23)].

References

[1] C. Amiot, Cluster categories for algebras of global dimension 2 and quivers with potential, Ann. Inst. Fourier 59 (2009) 2525–2590.

[2] I. Assem, T. Brüstle, G. Charbonneau-Jodoin, P-G. Plamondon, Gentle algebras arising from surface triangulations, Algebra Number Theory 4 (1) (2010) 201–229.

[3] A. Berenstein, S. Fomin, A. Zelevinsky, Cluster algebras. III. Upper bounds and double Bruhat cells, Duke Math. J. 126 (1) (2005) 1–52.

[4] A.B. Buan, R. Marsh, M. Reineke, I. Reiten, G. Todorov, Tilting theory and cluster combinatorics, Adv. Math. 204 (2006) 572–612.

[5] M.C.R. Butler, C.M. Ringel, Auslander–Reiten sequences with few middle terms, Comm. Algebra 15 (1987) 145–179.

[6] T. Brüstle, J. Zhang, On the cluster category of a marked surface without punctures, Algebra Number Theory 5 (4) (2011) 529–566.

[7] I. Canakci, R. Schiffler, Snake graph calculus and cluster algebras from surfaces, J. Algebra 382 (2013) 240–281.

[8] I. Canakci, R. Schiffler, Snake graph calculus and cluster algebras from surfaces II, Math. Z. 281 (1–2) (2015) 55–102.

[9] P. Caldero, F. Chapoton, R. Schiffler, Quivers with relations arising from clusters (an case), Trans. Amer. Math. Soc. 358 (3) (2006) 1347–1364.

[10] G. Cerulli Irelli, D. Labardini-Fragoso, Quivers with potentials associated to triangulated surfaces, Part III: tagged triangulations and cluster monomials, Compos. Math. 148 (6) (2012) 1833–1866.

[11] W.W. Crawley-Boevey, Maps between representations of zero-relation algebras, J. Algebra 126 (1989) 259–263.

[12] H. Derksen, J. Weyman, A. Zelevinsky, Quivers with potentials and their representations I: mutations, Selecta Math. (N.S.) 14 (2008) 59–119.

[13] A. Felikson, M. Shapiro, P. Tumarkin, Skew-symmetric cluster algebras of finite mutation type, J. Eur. Math. Soc. 14 (2012) 1135–1180.

[14] V. Fock, A. Goncharov, Moduli spaces of local systems and higher Teichmüller theory, Publ. Math. Inst. Hautes Études Sci. 103 (2006) 1–211.

[15] V. Fock, A. Goncharov, Cluster ensembles, quantization and the dilogarithm, Ann. Sci. Éc. Norm. Supér. (4) 42 (6) (2009) 865–930.

[16] S. Fomin, M. Shapiro, D. Thurston, Cluster algebras and triangulated surfaces. Part I: cluster complexes, Acta Math. 201 (2008) 83–146.

[17] S. Fomin, D. Thurston, Cluster algebras and triangulated surfaces. Part II: lambda lengths, preprint, arXiv:1210.5569.

[18] S. Fomin, A. Zelevinsky, Cluster algebras I: foundations, J. Amer. Math. Soc. 15 (2002) 497–529.

[19] S. Fomin, A. Zelevinsky, Cluster algebras II. Finite type classification, Invent. Math. 154 (1) (2003) 63–121.

[20] S. Fomin, A. Zelevinsky, Cluster algebras IV: coefficients, Compos. Math. 143 (2007) 112–164.

[21] C. Geiß, D. Labardini-Fragoso, J. Schröer, The representation type of Jacobian algebras, Adv. Math. 290 (2016) 364–452.

[22] O. Iyama, Y. Yoshino, Mutations in triangulated categories and rigid Cohen–Macauley modules, Invent. Math. 172 (2008) 117–168.

[23] B. Keller, D. Yang, Derived equivalences from mutations of quivers with potential, Adv. Math. 226 (3) (2011) 2118–2168.
[24] S. Koenig, B. Zhu, From triangulated categories to abelian categories-cluster tilting in a general framework, Math. Z. 258 (2008) 143–160.
[25] H. Krause, Maps between tree and band modules, J. Algebra 137 (1) (1991) 186–194.
[26] D. Labardini-Fragoso, Quivers with potentials associated to triangulated surfaces, Proc. Lond. Math. Soc. 98 (2009) 797–839.
[27] D. Labardini-Fragoso, Quivers with potentials associated to triangulated surfaces, Part II: arc representations, preprint, arXiv:0909.4100 [math.RT].
[28] D. Labardini-Fragoso, Quivers with potentials associated to triangulated surfaces, part IV: removing boundary assumptions, Selecta Math. (N.S.) 22 (1) (2016) 145–189.
[29] K. Lee, R. Schiffler, Positivity for cluster algebras, Ann. of Math. (2) 182 (1) (2015) 73–125.
[30] R. Marsh, Y. Palu, Coloured quivers for rigid objects and partial triangulations: the unpunctured case, Proc. Lond. Math. Soc. (3) 108 (2) (2014) 411–440.
[31] G. Musiker, R. Schiffler, Cluster expansion formulas and perfect matchings, J. Algebraic Combin. 32 (2) (2010) 187–209.
[32] G. Musiker, R. Schiffler, L. Williams, Positivity for cluster algebras from surfaces, Adv. Math. 227 (2011) 2241–2308.
[33] G. Musiker, R. Schiffler, L. Williams, Bases for cluster algebras from surfaces, Compos. Math. 149 no. 2 (2013) 217–263.
[34] G. Musiker, L. Williams, Matrix formulae and skein relations for cluster algebras from surfaces, Int. Math. Res. Not. 13 (2013) 2891–2944.
[35] P.-G. Plamondon, Cluster characters for cluster categories with infinite-dimensional morphism spaces, Adv. Math. 227 (1) (2011) 1–39.
[36] P.-G. Plamondon, Cluster algebras via cluster categories with infinite-dimensional morphism spaces, Compos. Math. 147 (2011) 1921–1954.
[37] J. Propp, The combinatorics of frieze patterns and Markoff numbers, preprint, arXiv:math/0511633.
[38] Y. Qiu, Y. Zhou, Cluster categories for marked surfaces: punctured case, preprint, arXiv:1311.0010.
[39] C. Ricke, On Jacobian algebras associated with the once-punctured torus, J. Pure Appl. Algebra 219 (11) (2015) 4998–5039.
[40] J. Schröer, Modules without self-extensions over gentle algebras, J. Algebra 216 (1) (1999) 178–189.
[41] J. Schröer, A. Zimmermann, Stable endomorphism algebras of modules over special biserial algebras, Math. Z. 244 (3) (2003) 515–530.
[42] J. Zhang, Y. Zhou, B. Zhu, Cotorsion pairs in the cluster category of a marked surface, J. Algebra 391 (2013) 209–226.